

WEAK COMPACTNESS AND SUMMABILITY

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Let $T = (c_{nm})_{n,m \in \mathbb{N}}$ be a method of summability. In agreement with the terminology employed in [2], we will say that T is almost regular* if the following conditions are satisfied:

- (i) $\lim_{n \rightarrow \infty} \sum_{m=1}^{\infty} c_{nm} = c$;
- (ii) $\lim_{n \rightarrow \infty} c_{nm} = c_m$ for every $m \in \mathbb{N}$;
- (iii) $c \neq \sum_{n=1}^{\infty} c_n$.

Here, of course, c and c_n ($n \in \mathbb{N}$) are finite, and the series are supposed to be convergent.

We will say that a subset of a Banach space has property \mathcal{S} if for every sequence in the subset there is an almost regular* summability method T such that the T -means of sequence converge weakly.

In [2] D. Waterman established a theorem which can be formulated as saying that the unit ball of a Banach space having property \mathcal{S} is weakly compact. We shall prove the following generalization of this result.

Theorem. *If a bounded subset of a Banach space has property \mathcal{S} , then it is weakly relatively compact.*

Proof. Suppose that a bounded subset A of a Banach space E has property \mathcal{S} . Should not the weak closure of A be weakly compact, then, by a result of Kadec and Pełczyński [1], there is a basic sequence $(x_n)_{n \in \mathbb{N}}$ in A for which the origin is not a weak cluster point. By passing to a subsequence if necessary, we can assume that there exists x^* in E^* , the dual space of E , such that

$$\sum_{n=1}^{\infty} |x^*(x_n) - 1| < \infty.$$

Of course, $D = \inf\{\|x_n\| : n \in \mathbb{N}\} > 0$. Let C be a positive number such that, for every choice of $n, m \in \mathbb{N}$ with $n < m$ and scalars λ_i ($1 \leq i \leq m$), we have

$$\left\| \sum_{i=1}^n \lambda_i x_i \right\| \leq C \left\| \sum_{i=1}^m \lambda_i x_i \right\|.$$

Then

$$\begin{aligned} \left| \sum_{i=1}^m \lambda_i \right| &\leq \left| \sum_{i=1}^m \lambda_i (x^*(x_i) - 1) \right| + \left| \sum_{i=1}^m \lambda_i x^*(x_i) \right| \\ &\leq D^{-1} \max\{\|\lambda_i x_i\| : 1 \leq i \leq m\} \sum_{i=1}^m |x^*(x_i) - 1| \\ &\quad + \|x^*\| \left\| \sum_{i=1}^m \lambda_i x_i \right\| \\ &\leq \left(2CD^{-1} \sum_{i=1}^{\infty} |x^*(x_i) - 1| + \|x^*\| \right) \left\| \sum_{i=1}^m \lambda_i x_i \right\|. \end{aligned}$$

This inequality jointly with the Hahn-Banach theorem shows that there is z^* in E^* such that

$$(1) \quad z^*(x_n) = 1$$

for all $n \in \mathbb{N}$.

According to our hypothesis, there is an almost regular* summability method $T = (c_{nm})_{n,m \in \mathbb{N}}$ and a point x in the closed linear span of $\{x_n : n \in \mathbb{N}\}$ such that the T -means of $(x_n)_{n \in \mathbb{N}}$ converge weakly to x . Let $(x_n^*)_{n \in \mathbb{N}}$ be a sequence in E^* such that

$$x_n^*(x_m) = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

In view of (ii), for each $n \in \mathbb{N}$, the T -means of $(x_n^*(x_m))_{m \in \mathbb{N}}$ converge to c_n . Hence

$$x = \sum_{n=1}^{\infty} c_n x_n$$

and further, in view of (1),

$$z^*(x) = \sum_{n=1}^{\infty} c_n.$$

On the other hand, by virtue of (i) and (1), the T -means of $(z^*(x_n))_{n \in \mathbb{N}}$ converge to c . This and the above equality imply

$$\sum_{n=1}^{\infty} c_n = c,$$

which contradicts (iii).

The proof is complete.

REFERENCES

1. M. I. Kadec and A. Pełczyński, *Basic sequences, bi-orthogonal systems and norming sets in Banach and Fréchet spaces*, *Studia Math.* **25** (1965), 297–323 (Russian).
2. D. Waterman, *Reflexivity and summability*, *Studia Math.* **32** (1969), 61–63.

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