ON HOMOMORPHISMS FROM SEMIGROUP ALGEBRAS: GENERATION, CHARACTERS, AND ISOLATED POINTS

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Abstract. We investigate the problem of identifying isolated points in spaces of continuous homomorphisms from ordered AL-algebras to normed algebras. Building on recent work of Bobrowski and the author [Dissertationes Math. (Rozprawy Mat.) **577** (2022), 1–69], we consider homomorphisms from semigroup algebras associated with non-locally-null, Haar measurable subsemigroups of locally compact groups. A key novelty is the removal of the assumption that the semigroup algebra admits an approximate identity of bound 1, which was essential in prior work. Our main result shows that for any continuous character σ on a suitable semigroup S, the homomorphism $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma \colon L^1(S) \to \mathfrak{A},\ f \mapsto \mathscr{L}_\sigma(f) e_{\mathfrak{A}}$, is isolated in the space of all continuous homomorphisms from $L^1(S)$ to a unital normed algebra $\mathfrak{A}$. Here $L^1(S)$ denotes the semigroup algebra of S, $e_{\mathfrak{A}}$ is the unit of $\mathfrak{A}$, and $\mathscr{L}_\sigma$ is the character on $L^1(S)$ determined by σ. This follows from a quantitative rigidity theorem: any continuous homomorphism within operator-norm distance less than 1 from $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$ must coincide with it. Along the way, we develop a general framework for constructing and analysing such homomorphisms, including integral and associated representations, which serve not only to establish the main result, but also to characterise characters on semigroup algebras.

1. Introduction

This paper addresses the problem of identifying isolated points in spaces of continuous homomorphisms from ordered AL-algebras to normed algebras, where all homomorphisms within a given space share a common domain and a common codomain. The same problem was recently explored in [5], a work that continues a long-standing line of research on isolability in spaces of operators defined on a single Banach space and subject to various structural conditions. Prior to [5], isolated points were investigated in spaces of operator-valued solutions to a range of functional equations, including (i) strongly continuous semigroups and cosine families of operators [3], [4], [6], [9], [10], [16], [17], [31]; (ii) integrated semigroups and operator sine functions [2]; and (iii) operator-valued solutions of fractional evolution equations [19]. The present article builds on the study in [5] and focuses on isolated points in spaces of continuous homomorphisms from specific ordered AL-algebras—namely, semigroup algebras associated with non-locally-null, Haar measurable subsemigroups of locally compact groups.

A key assumption in [5] for identifying isolated points in spaces of continuous homomorphisms from semigroup algebras was that each semigroup algebra under consideration admits an approximate identity of bound 1. In the present paper, we show that this assumption can be relaxed: the results of [5] remain valid for semigroup algebras that do not necessarily possess an approximate identity of bound 1.

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To give a more concrete idea of our contribution, suppose that G is a locally compact group, S is a non-locally-null, Haar measurable subsemigroup of G, and $\mathfrak{A}$ is a unital normed algebra with a unit $e_{\mathfrak{A}}$. Let $L^1(S)$ denote the semigroup algebra of S based on a left Haar measure m_G on G. Given a continuous character σ on S, let $\mathscr{L}_\sigma$ denote the corresponding character on $L^1(S)$ defined by

$$\mathscr{L}_\sigma(f) = \int_S f(s)\sigma(s)\,\mathrm{d}m_G(s) \quad (f \in L^1(S)).$$

Our main result states that, if σ is a continuous character on S, then the homomorphism

$$e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma \colon L^1(S) \to \mathfrak{A}, \quad f \mapsto \mathscr{L}_\sigma(f)e_{\mathfrak{A}},$$

is isolated in the space of all continuous homomorphisms from $L^1(S)$ to $\mathfrak{A}$. We shall in fact prove a stronger, quantitative version of this result: if σ is a continuous character on S and if $H\colon L^1(S) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$, then $H = e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$. This generalises a similar result in [5], where an additional assumption was imposed to ensure that $L^1(S)$ possesses an approximate identity of bound 1.

While this generalisation is the primary focus of the article, several other topics will naturally arise in the course of the development. We discuss these briefly as we outline the structure of the paper below.

In Section 2, we recall or introduce some notation, definitions, and background results about semigroups and their algebras.

To meaningfully discuss isolated points in spaces of homomorphisms from semigroup algebras, it is essential to have a sufficiently broad class of such homomorphisms available. Section 3 is devoted to the construction of semigroup algebra homomorphisms. We show that a broad variety of such homomorphisms can be realised as representations of semigroup algebras on Banach spaces, each defined by an integral involving a uniformly bounded semigroup of operators. We also exhibit examples of semigroup algebra representations that cannot be expressed in this integral form.

In Section 4, we introduce the notion of a representation associated with a given homomorphism from a semigroup algebra, acting on the range space of that homomorphism. We further show that each associated representation admits an integral representation of the type described above—a key fact for subsequent applications.

In Section 5, we use associated representations to characterise the characters on semigroup algebras. Although this characterisation is of independent interest, its inclusion here is motivated by its role in situating the main result of the paper in its proper context.

Finally, in Section 6, we employ associated representations to establish the paper's main result together with some of its consequences.

2. Preliminaries

In this section we shall recall or introduce some notation, definitions, and results.

2.1. **Basic notation.** We shall use the following standard notation: $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers; $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ is the set of integers; $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ is the set of non-negative integers; $\mathbb{R}$ is the real line; $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$ is the non-negative half-line; $\mathbb{C}$ is the complex plane; $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane; $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed unit disc in the complex plane.

The characteristic function of a subset A of a set is denoted by χ_A.

Let X be a normed space (always over the complex field $\mathbb{C}$). We denote by $X_{[1]}$ the closed unit ball in X. By $\mathscr{L}(X)$ we denote the normed algebra of all bounded linear operators on X; it is a Banach algebra when X is a Banach space.

2.2. **Algebras.** Let $\mathfrak{A}$ be an algebra. An *idempotent* in $\mathfrak{A}$ is an element $p \in \mathfrak{A}$ such that $p^2 = p$.

A *character* on a complex algebra $\mathfrak{A}$ is a homomorphism from $\mathfrak{A}$ onto $\mathbb{C}$. Every character on a Banach algebra is continuous and has norm at most 1.

Given a complex algebra $\mathfrak{A}$, the set of all characters on $\mathfrak{A}$ is denoted by $\Delta(\mathfrak{A})$ and is called the *character space* of $\mathfrak{A}$.

Let $\mathfrak{A}$ be a normed algebra. A net $\{e_\iota\}_{\iota\in I}$ in $\mathfrak{A}$ is called a *left* (respectively, *right*, or *two-sided*) *approximate identity* for $\mathfrak{A}$ if

$$\lim_{\iota\in I} \|x - e_\iota x\| = 0$$

(respectively,

$$\lim_{\iota\in I} \|x - x e_\iota\| = 0$$

or

$$\lim_{\iota\in I} (\|x - e_\iota x\| + \|x - x e_\iota\|) = 0)$$

for every $x \in \mathfrak{A}$. Such a net is said to be *bounded* if there exists $K > 0$ such that $\|e_\iota\| \le K$ for all $\iota \in I$; the least such K, that is, $\sup_{\iota\in I} \|e_\iota\|$, is called the *bound* of $\{e_\iota\}_{\iota\in I}$. If $\mathfrak{A}$ is commutative, one simply speaks of a (bounded) approximate identity.

2.3. **Semigroups and semi-characters.** Let S be a semigroup, with the product of two elements denoted by juxtaposition. For subsets A and B of S, we set

$$AB = \{st : s \in A,\ t \in B\}.$$

A non-empty subset I of S is a *left ideal* in S if $SI \subset I$, and a *right ideal* if $IS \subset I$; a subset is an ideal if it is both a left and a right ideal.

Let S be a semigroup. A *semi-character* on S is a map $\zeta\colon S \to \overline{\mathbb{D}}$, not identically zero, such that

$$\zeta(st) = \zeta(s)\zeta(t) \quad (s, t \in S).$$

A *character* on S is a semi-character on S taking values in the unit circle $\mathbb{T}$. The function identically equal to 1 on S is the *trivial character* on S, also called the *augmentation character*. In general, the trivial character may be the only semi-character on S.

Let S be a topological semigroup, i. e., a semigroup equipped with a Hausdorff topology such that the mapping $(s, t) \mapsto st$ from $S \times S$ into S is continuous. We denote by $\widehat{S}$ the set of all continuous semi-characters on S, and by $\widehat{S}_{\mathrm{u}}$ the set of all continuous characters on S.

2.4. **Semigroup algebras.** Let G be a locally compact group, and let m_G denote a fixed but arbitrary left Haar measure on G. Let A be a Haar measurable subset of G that is not locally null with respect to m_G. We write $L^1(A)$ for the space of complex-valued, Haar integrable functions on A, where functions that agree almost everywhere are identified. This space is equipped with the norm

$$\|f\|_1 = \int_A |f(s)|\,\mathrm{d}m_G(s) \quad (f \in L^1(A)).$$

The requirement that A is not locally null ensures that $L^1(A)$ is non-trivial, i. e., contains more than just the zero function. In the case where G and A are discrete, we write $\ell^1(A)$ for $L^1(A)$.

The space $L^1(G)$ is a Banach algebra under the convolution product

$$(f \star g)(s) = \int_G f(t)g(t^{-1}s)\,\mathrm{d}m_G(t) \quad (f, g \in L^1(G),\ \text{a.e. } s \in G),$$

where "a. e." stands for "m_G-almost every." The algebra $L^1(G)$ is called the *group algebra* of G.

Let S be a non-locally-null, Haar measurable subsemigroup of G. The space $L^1(S)$ is a Banach algebra for the convolution product defined implicitly by

$$\int_S h(f \star g)\,\mathrm{d}m_G = \int_{S\times S} h(st)f(s)g(t)\,\mathrm{d}(m_G \times m_G)(s,t) \qquad (f, g \in L^1(S),\ h \in L^\infty(S)),$$

where $L^\infty(S)$ denotes the Banach space of complex-valued, essentially bounded, Haar measurable functions on S, identified up to equality locally almost everywhere. An equivalent description of the convolution product is as follows. For a Haar measurable function h on S, let $\tilde{h}$ denote the function on G that agrees almost everywhere with h on S and vanishes almost everywhere outside S; such a function is uniquely determined up to equality almost everywhere. Then, for $f, g \in L^1(S)$, the convolution $f \star g$ is the restriction of $\tilde{f} \star \tilde{g}$ to S. More explicitly,

$$(f \star g)(s) = \int_{S\cap sS^{-1}} f(t)g(t^{-1}s)\,\mathrm{d}m_G(t) \quad (\text{a. e. } s \in S), \tag{2.1}$$

where

$$sS^{-1} = \{st^{-1} : t \in S\}.$$

The algebra $L^1(S)$ is called the *semigroup algebra* of S.

2.5. **Density points.** Let G be a locally compact group. A *density point* of a Haar measurable subset A of G is an element s in G such that every open neighbourhood of s in G intersects A in a set of positive Haar measure. We denote the set of all density points of A by $D(A)$. Simon [33] proved that, for any Haar measurable subset A of G, the following properties hold:

(i) $D(A)$ is closed;
(ii) $A^\circ \subset D(A) \subset \overline{A}$, where A° and $\overline{A}$ are the interior and the closure of A, respectively;
(iii) $A \setminus D(A)$ is locally null.

Moreover, Simon showed that, if S is a Haar measurable subsemigroup of G, then $D(S)$ is also a semigroup.

Let S be a non-locally-null, Haar measurable subsemigroup of G. Set

$$S_D = S \cap D(S).$$

Then S_D is a subsemigroup of S and, in fact, an ideal in S. We call S_D the *density subsemigroup* of S. Since S_D has full (Haar) measure in S, it follows that $L^1(S) = L^1(S_D)$.

As a simple illustration of the above concepts, consider $\mathbb{R}$ as a group under addition, equipped with its usual topology. Let

$$S = \{0\} \cup (1, \infty).$$

Then S is a Lebesgue measurable subsemigroup of $\mathbb{R}$, and

$$D(S) = [1, \infty) \quad \text{and} \quad S_D = (1, \infty).$$

Returning to the general case of G and S, in the following result—and later—we shall use some specific functions that are defined on S.

Definition 2.1. Let $s \in S_D$. Let $\{V_\iota\}_{\iota\in I}$ be a base of precompact (i.e., with compact closure) open neighbourhoods of s in G, directed downward by inclusion (so that I is a directed set with a partial ordering $\preceq$ such that, for any $\iota, \kappa \in I$, $\iota \preceq \kappa$ if and only if $V_\kappa \subset V_\iota$). For each $\iota \in I$, define

$$f_\iota = \frac{1}{m_G(V_\iota \cap S)} \chi_{V_\iota \cap S};$$

since $s \in D(S)$, we have $m_G(V_\iota \cap S) > 0$, so f_ι is well defined.

The family $\{f_\iota\}_{\iota\in I}$ is thus a net of non-negative, compactly supported, Haar-integrable functions over S, with supports that "tend" to $\{s\}$, and such that $\int_S f_\iota \,\mathrm{d}m_G = 1$ for each $\iota \in I$.

Lemma 2.2. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G, and let X be a Banach space.*

(i) *Suppose that A and B are subsets of G, each containing S_D, and that $f\colon A \to X$ and $g\colon B \to X$ are continuous functions that agree locally almost everywhere on $A \cap B$. Then $f = g$ on S_D.*

(ii) *Suppose that A is a subset of S containing S_D, and that $f\colon A \to X$ is a bounded continuous function. Then*

$$\operatorname*{ess\,sup}_{s\in A} \|f(s)\| := \inf\left\{ \sup_{s\in A\setminus N} \|f(s)\| : N \text{ is locally null} \right\} = \sup_{s\in S_D} \|f(s)\|.$$

Proof. (i) Let $s \in S_D$, and let $\{f_\iota\}_{\iota\in I}$ be as defined in Definition 2.1. Then, for each $\iota \in I$, both $f_\iota f$ and $f_\iota g$ are Haar integrable. Moreover, since f and g agree locally almost everywhere on $A \cap B$,

$$\int_{A\cap B} f_\iota(t) f(t)\,\mathrm{d}m_G(t) = \int_{A\cap B} f_\iota(t) g(t)\,\mathrm{d}m_G(t) \quad (\iota \in I). \tag{2.2}$$

Since A and B are both locally of full measure in S, $A \cap B$ is also locally of full measure in S. Hence, for each $\iota \in I$,

$$m_G(V_\iota \cap S) = m_G(V_\iota \cap A \cap B)$$

and thus

$$f_\iota = \frac{1}{m_G(V_\iota \cap A \cap B)} \chi_{V_\iota \cap S}.$$

Now, since f and g are continuous, a standard argument yields

$$\lim_{\iota\in I} \int_{A\cap B} f_\iota(t) f(t)\,\mathrm{d}m_G(t) = f(s) \quad \text{and} \quad \lim_{\iota\in I} \int_{A\cap B} f_\iota(t) g(t)\,\mathrm{d}m_G(t) = g(s).$$

Therefore, by (2.2), we conclude that $f(s) = g(s)$.

(ii) Since $A \setminus S_D$ is locally null, we have

$$\operatorname*{ess\,sup}_{s\in A} \|f(s)\| = \operatorname*{ess\,sup}_{s\in S_D} \|f(s)\|.$$

As $\operatorname{ess\,sup}_{s\in S_D} \|f(s)\| \le \sup_{s\in S_D} \|f(s)\|$, it follows that

$$\operatorname*{ess\,sup}_{s\in A} \|f(s)\| \le \sup_{s\in S_D} \|f(s)\|.$$

For the reverse inequality, fix $s \in S_D$, and let $\{f_\iota\}_{\iota\in I}$ be as before. Since A is locally of full measure in S, for each $\iota \in I$ we have

$$m_G(V_\iota \cap S) = m_G(V_\iota \cap A)$$

and hence

$$f_\iota = \frac{1}{m_G(V_\iota \cap A)} \chi_{V_\iota \cap S}.$$

Now, the continuity of $t \mapsto \|f(t)\|$, together with the above identity, implies that

$$\lim_{\iota \in I} \int_A f_\iota(t)\, \|f(t)\| \,\mathrm{d}m_G(t) = \|f(s)\|.$$

Moreover, for each $\iota \in I$,

$$\int_A f_\iota(t)\, \|f(t)\| \,\mathrm{d}m_G(t) \le \|f_\iota\|_1 \operatorname*{ess\,sup}_{t \in A} \|f(t)\| = \operatorname*{ess\,sup}_{s \in A} \|f(s)\|.$$

Hence

$$\|f(s)\| \le \operatorname*{ess\,sup}_{s \in A} \|f(s)\|.$$

Taking the supremum over $s \in S_D$ yields the desired inequality. □

3. Representations of semigroup algebras

A broad variety of homomorphisms from semigroup algebras arises as representations of these algebras on Banach spaces. In this section, we show that a class of such representations originates from uniformly bounded semigroups of operators on Banach spaces. Each member of this class can be expressed as an integral with a semigroup of operators appearing in the integrand. However, this construction is not exhaustive: we present a family of representations that cannot be expressed in the integral form just mentioned.

3.1. **Algebra-valued semigroups.** Let S be a semigroup. In the case where S is unital, the neutral element of S is denoted by e_S. Let $\mathfrak{A}$ be a unital algebra, with a unit $e_{\mathfrak{A}}$. An $\mathfrak{A}$-valued family $\{\mathscr{S}(s)\}_{s \in S}$ is a *semigroup* in $\mathfrak{A}$ if:

(i) $\mathscr{S}(s)\mathscr{S}(t) = \mathscr{S}(st)$ for all $s, t \in S$;
(ii) $\mathscr{S}(e_S) = e_{\mathfrak{A}}$ whenever S is unital.

An $\mathscr{L}(X)$-valued semigroup, where X is a non-zero normed space, is called a semigroup on X.

Let A be a topological space. A family $\{F(a)\}_{a \in A}$ of bounded linear operators on a normed space X is *strongly continuous* if, for each $x \in X$, the function $A \ni a \mapsto F(a)x \in X$ is continuous in norm.

Let G be a locally compact group. Suppose that S is a non-locally-null, Haar measurable subsemigroup of G. Given $s \in S$, we denote by S_s the operator of *right shift* by s on $L^1(S)$, defined by

$$(S_s f)(t) = \begin{cases} f(s^{-1}t) & \text{for a. e. } t \in sS, \\ 0 & \text{for a. e. } t \in S \setminus sS \end{cases} \qquad (f \in L^1(S)).$$

It is straightforward to verify that the family $\{S_s\}_{s \in S}$ is a strongly continuous semigroup on $L^1(S)$ and, moreover, that

$$S_s(f \star g) = S_s f \star g \tag{3.1}$$

holds for all $s \in S$ and all $f, g \in L^1(S)$.

3.2. **Semigroup-representable representations.** Let $\mathfrak{A}$ be an algebra and let X be a normed space. A *representation* of $\mathfrak{A}$ on X is a homomorphism from $\mathfrak{A}$ into $\mathscr{L}(X)$.

Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G, and let X be a Banach space. Let $\mathscr{S} = \{\mathscr{S}(s)\}_{s \in S_D}$ be a uniformly bounded, strongly continuous semigroup on X, indexed by the density subsemigroup S_D of S. Then, for each $f \in L^1(S)$ and each $x \in X$, the function

$$S_D \to X, \quad s \mapsto f(s)\mathscr{S}(s)x,$$

is Haar measurable and is dominated in norm by the Haar integrable function $s \mapsto |f(s)| \sup_{t\in S_D} \|\mathscr{S}(t)\| \, \|x\|$. Therefore $s \mapsto f(s)\mathscr{S}(s)x$ is Bochner integrable, and we may define an element $H(f)x$ of X by

$$H(f)x = \int_{S_D} f(s)\mathscr{S}(s)x \, \mathrm{d}m_G(s),$$

where the integral is understood in the Bochner sense. This element satisfies

$$\|H(f)x\| \leq \|f\|_1 \sup_{s\in S_D} \|\mathscr{S}(s)\| \, \|x\|.$$

For each $f \in L^1(S)$, the mapping

$$H(f)\colon X \to X, \quad x \mapsto H(f)x,$$

is a linear operator in $\mathscr{L}(X)$, with norm at most $\|f\|_1 \sup_{s\in S_D} \|\mathscr{S}(s)\|$. Consequently, the linear mapping

$$H\colon L^1(S) \to \mathscr{L}(X), \quad f \mapsto H(f),$$

is bounded, with norm at most $\sup_{s\in S_D} \|\mathscr{S}(s)\|$.

Definition 3.1. We say that the mapping H above is the *integrated form* of $\mathscr{S}$ (see, e. g., [18, Section VIII.13.9, p. 858] or [35, Proposition 2.23] for the use of this terminology in the existing literature).

In a classical construction, semigroups of operators are used to define homomorphisms from $L^1(\mathbb{R}^+)$, where $\mathbb{R}^+$ is viewed as a subsemigroup of $\mathbb{R}$, and $\mathbb{R}$ is considered as a group under addition, equipped with its usual topology (see, e. g., [1, 7.4.46, p. 262]). The following theorem extends this idea to homomorphisms from semigroup algebras belonging to a broader class that is relevant to this study.

Theorem 3.2. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G, and let X be a Banach space. Let $\mathscr{S} = \{\mathscr{S}(s)\}_{s\in S_D}$ be a uniformly bounded, strongly continuous semigroup on X, and let $H\colon L^1(S) \to \mathscr{L}(X)$ be the integrated form of $\mathscr{S}$. Then H is a continuous homomorphism, and $\|H\| = \sup_{s\in S_D} \|\mathscr{S}(s)\|$.*

Proof. To show that H is a homomorphism, we need only demonstrate that H is multiplicative, that is,

$$H(f \star g) = H(f)H(g) \tag{3.2}$$

for all $f, g \in L^1(S)$.

Let $f, g \in L^1(S)$ and $x \in X$. Since $L^1(S_D) = L^1(S)$, the expression for the convolution $f \star g$ in (2.1) can be rewritten as

$$(f \star g)(s) = \int_{S_D \cap sS_D^{-1}} f(t)g(t^{-1}s) \, \mathrm{d}m_G(t) \quad (\text{a. e. } s \in S_D).$$

Thus, we have

$$\begin{aligned} H(f \star g)x &= \int_{S_D} (f \star g)(s)\mathscr{S}(s)x \, \mathrm{d}m_G(s) \\ &= \int_{S_D} \left[\int_{S_D \cap sS_D^{-1}} f(t)g(t^{-1}s) \, \mathrm{d}m_G(t) \right] \mathscr{S}(s)x \, \mathrm{d}m_G(s). \end{aligned} \tag{3.3}$$

For a Haar measurable function h on S_D, whether scalar- or vector-valued, let $\tilde{h}$ denote the function on G that agrees almost everywhere with h on S_D and vanishes almost everywhere outside S_D (slightly modifying the earlier definition based on S). Since, for almost every $s \in S_D$,

$$\int_{S_D \cap s S_D^{-1}} f(t) g(t^{-1}s)\, \mathrm{d}m_G(t) = \int_{S_D} f(t)\tilde{g}(t^{-1}s)\, \mathrm{d}m_G(t),$$

we may apply Fubini's theorem for locally compact spaces (cf. [12, Theorem A.4.12] or [20, §13]) to obtain

$$\begin{aligned}
(3.4)\quad \int_{S_D} \left[\int_{S_D \cap s S_D^{-1}} f(t) g(t^{-1}s)\, \mathrm{d}m_G(t) \right] \mathscr{S}(s) x \, \mathrm{d}m_G(s) \\
&= \int_{S_D} \left[\int_{S_D} f(t) \tilde{g}(t^{-1}s)\, \mathrm{d}m_G(t) \right] \mathscr{S}(s) x\, \mathrm{d}m_G(s) \\
&= \int_{S_D \times S_D} f(t)\tilde{g}(t^{-1}s) \mathscr{S}(s) x \, \mathrm{d}m_{G\times G}(t,s) \\
&= \int_{S_D \times S_D} f(t)\tilde{g}(t^{-1}s) \mathscr{S}(t)\tilde{\mathscr{S}}(t^{-1}s) x \, \mathrm{d}m_{G\times G}(t,s) \\
&= \int_{S_D} f(t) \left[\int_{S_D} \tilde{g}(t^{-1}s) \mathscr{S}(t)\tilde{\mathscr{S}}(t^{-1}s) x\, \mathrm{d}m_G(s) \right] \mathrm{d}m_G(t).
\end{aligned}$$

By the interchangeability of the Bochner integral with bounded linear operators (cf. [21, Theorem 3.7.12] or [37, p. 134, Corollary 2]), if $t \in S_D$, then

$$\int_{S_D} \tilde{g}(t^{-1}s)\mathscr{S}(t)\tilde{\mathscr{S}}(t^{-1}s) x\, \mathrm{d}m_G(s) = \mathscr{S}(t) \left(\int_{S_D} \tilde{g}(t^{-1}s)\tilde{\mathscr{S}}(t^{-1}s)x \, \mathrm{d}m_G(s) \right).$$

Now, by the left invariance of m_G,

$$\int_{S_D} \tilde{g}(t^{-1}s)\tilde{\mathscr{S}}(t^{-1}s) x\, \mathrm{d}m_G(s) = \int_{t^{-1}S_D} \tilde{g}(u)\tilde{\mathscr{S}}(u) x \, \mathrm{d}m_G(u).$$

But $t^{-1}S_D \supset S_D$ (because S_D is an ideal), and $\tilde{g}$ and $\tilde{\mathscr{S}}$ vanish outside S_D and, in particular, on $t^{-1}S_D \setminus S_D$. Hence

$$\begin{aligned}
\int_{t^{-1}S_D} \tilde{g}(u)\tilde{\mathscr{S}}(u)x\, \mathrm{d}m_G(u) &= \int_{S_D} \tilde{g}(u)\tilde{\mathscr{S}}(u)x\, \mathrm{d}m_G(u) \\
&= \int_{S_D} g(u)\mathscr{S}(u)x\, \mathrm{d}m_G(u) = H(g)x.
\end{aligned}$$

We conclude that

$$\int_{S_D} \tilde{g}(t^{-1}s)\mathscr{S}(t)\tilde{\mathscr{S}}(t^{-1}s)x\, \mathrm{d}m_G(s) = \mathscr{S}(t)H(g)x.$$

Combining this with (3.3) and (3.4), we find that

$$\begin{aligned}
H(f \star g)x &= \int_{S_D} f(t) \left[\int_{S_D} \tilde{g}(t^{-1}s)\mathscr{S}(t)\tilde{\mathscr{S}}(t^{-1}s)x\, \mathrm{d}m_G(s) \right] \mathrm{d}m_G(t) \\
&= \int_{S_D} f(t)\mathscr{S}(t)H(g)x\, \mathrm{d}m_G(t) = H(f)H(g)x.
\end{aligned}$$

We see that (3.2) holds, as required.

As noted before the statement of the theorem, H is continuous, with $\|H\| \leq \sup_{s\in S_D} \|\mathscr{S}(s)\|$. We now show that $\|H\| = \sup_{s \in S_D} \|\mathscr{S}(s)\|$.

Let $x \in X$. Applying [14, Chap. III, Sect. 1, Lemma 4] or [15, 5.4] to the map

$$L^1(S) \to X, \quad f \mapsto \int_{S_D} f(s)\mathscr{S}(s)x\, \mathrm{d}m_G(s),$$

we conclude that

$$\sup_{f\in L^1(S)_{[1]}} \|H(f)x\| = \operatorname*{ess\,sup}_{s\in S_D} \|\mathscr{S}(s)x\|.$$

By Lemma 2.2,

$$\operatorname*{ess\,sup}_{s\in S_D} \|\mathscr{S}(s)x\| = \sup_{s\in S_D} \|\mathscr{S}(s)x\|.$$

Thus

$$\sup_{f\in L^1(S)_{[1]}} \|H(f)x\| = \sup_{s\in S_D} \|\mathscr{S}(s)x\|.$$

Consequently,

$$\begin{aligned}\|H\| &= \sup_{f\in L^1(S)_{[1]}} \sup_{x\in X_{[1]}} \|H(f)x\| = \sup_{x\in X_{[1]}} \sup_{f\in L^1(S)_{[1]}} \|H(f)x\| \\ &= \sup_{x\in X_{[1]}} \sup_{s\in S_D} \|\mathscr{S}(s)x\| = \sup_{s\in S_D} \sup_{x\in X_{[1]}} \|\mathscr{S}(s)x\| = \sup_{s\in S_D} \|\mathscr{S}(s)\|. \qquad \square\end{aligned}$$

Motivated by Theorem 3.2, we introduce the following definition.

Definition 3.3. Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G, and let X be a Banach space. We say that a continuous representation $H\colon L^1(S) \to \mathscr{L}(X)$ is *semigroup-representable* if there exists a uniformly bounded, strongly continuous semigroup $\mathscr{S} = \{\mathscr{S}(s)\}_{s\in S_D}$ on X such that H is the integrated form of $\mathscr{S}$.

In general, the abundance of operator semigroups ensures the existence of many semigroup-representable representations of semigroup algebras. However, as we shall demonstrate next, there exist choices of G, S, and X for which $L^1(S)$ admits continuous representations on X that are not semigroup-representable.

3.3. **A family of non-semigroup-representable representations.** Let ℓ^∞ denote the space of all bounded, complex-valued sequences, endowed with the norm

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| \quad (x = \{x_n\}_{n\in\mathbb{N}} \in \ell^\infty).$$

To show that non-semigroup-representable representations exist, we construct a family of continuous representations of $L^1(S)$ on X that fail to be semigroup-representable in the specific case where $G = \mathbb{R}$ with the usual topology, $S = S_D = \mathbb{R}^+$, and $X = \ell^\infty$. Each representation in this family is associated with a distinct sequence of real numbers tending to infinity, so the family is uncountably infinite.

For each $f \in L^1(\mathbb{R}^+)$, the *half-line Fourier transform* of f is the function $\lambda \mapsto \mathscr{F}_\lambda(f)$ on $\mathbb{R}$, defined by

$$\mathscr{F}_\lambda(f) = \int_0^\infty f(s)\mathrm{e}^{-i\lambda s}\,\mathrm{d}s \quad (\lambda\in\mathbb{R}).$$

It is well known that, for each $\lambda \in \mathbb{R}$, the mapping

$$\mathscr{F}_\lambda\colon L^1(\mathbb{R}^+) \to \mathbb{C}, \quad f \mapsto \mathscr{F}_\lambda(f),$$

is a character on $L^1(\mathbb{R}^+)$. Characters of this form do not exhaust the character space of $L^1(\mathbb{R}^+)$ (cf. [12, Theorem 4.7.27] or [28, p. 531]).

Given a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ and a positive integer n, we write $(x)_n$ for the nth element of x; that is, $(x)_n = x_n$.

Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. For each $f \in L^1(\mathbb{R}^+)$, define a linear operator $H_\omega(f)$ on ℓ^∞ by

$$(H_\omega(f)x)_n = \mathscr{F}_{\omega_n}(f)x_n \quad (x = \{x_n\}_{n\in\mathbb{N}} \in \ell^\infty).$$

Since

$$\left|\mathscr{F}_{\omega_n}(f)\right| \le \|f\|_1 \quad (n\in\mathbb{N}),$$

the operator $H_\omega(f)$ is bounded, with norm at most $\|f\|_1$. Consequently, the linear mapping

$$H_\omega\colon L^1(\mathbb{R}^+) \to \mathscr{L}(\ell^\infty), \quad f \mapsto H_\omega(f),$$

is bounded, with norm at most 1. Moreover, since $\mathscr{F}_{\omega_n}$ is a character on $L^1(\mathbb{R}^+)$ for each $n \in \mathbb{N}$, it follows that H_ω is a homomorphism; specifically, it is a representation of $L^1(\mathbb{R}^+)$ on ℓ^∞.

Theorem 3.4. *Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers tending to infinity. Then the representation $H_\omega\colon L^1(\mathbb{R}^+) \to \mathscr{L}(\ell^\infty)$ is not semigroup-representable.*

We shall need a couple of lemmas to prove this theorem.

Let c_0 denote the space of all complex-valued sequences that converge to zero.

Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. For each $t \in \mathbb{R}$, let $\mathscr{V}_\omega(t)$ be the linear operator on ℓ^∞ defined by

$$(\mathscr{V}_\omega(t)x)_n = \mathrm{e}^{-\mathrm{i}\omega_n t} x_n \quad (x \in \ell^\infty,\ x = \{x_n\}_{n\in\mathbb{N}}).$$

Then $\mathscr{V}_\omega(t)$ is bounded with norm 1.

Lemma 3.5. *Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers tending to infinity. Then, for each $x \in \ell^\infty \setminus c_0$, the map $\mathbb{R} \ni t \mapsto \mathscr{V}_\omega(t)x \in \ell^\infty$ is nowhere continuous.*

Proof. Fix $x \in \ell^\infty \setminus c_0$. Then there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ with $n_k \to \infty$ such that

$$c := \inf_{k\in\mathbb{N}} |x_{n_k}| > 0.$$

Since $\omega_n \to \infty$ as $n \to \infty$, there exists $k_0 \in \mathbb{N}$ such that $\omega_{n_k} > 0$ for all $k \geq k_0$. For each $k \geq k_0$, set $t_k = \pi/\omega_{n_k}$. Then $t_k \to 0$ as $k \to \infty$. Moreover, for any $t \in \mathbb{R}$ and any $k \geq k_0$,

$$\|\mathscr{V}_\omega(t+t_k)x - \mathscr{V}_\omega(t)x\|_\infty \geq |(\mathscr{V}_\omega(t+t_k)x)_{n_k} - (\mathscr{V}_\omega(t)x)_{n_k}| = 2|x_{n_k}| \geq 2c.$$

Hence $t \mapsto \mathscr{V}_\omega(t)x$ is nowhere continuous. □

Lemma 3.6. *Let $\omega = \{\omega_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Let $x = \{x_n\}_{n\in\mathbb{N}} \in \ell^\infty$ and suppose that $h_{x,\omega}\colon \mathbb{R}^+ \to \ell^\infty$ is continuous and satisfies*

$$H_\omega(f)x = \int_0^\infty f(s) h_{x,\omega}(s)\,\mathrm{d}s$$

for each $f \in L^1(\mathbb{R}^+)$. Then $h_{x,\omega}(t) = \mathscr{V}_\omega(t)x$ for each $t \in \mathbb{R}^+$.

Proof. Fix $t \in \mathbb{R}^+$, and let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions in $L^1(\mathbb{R}^+)$ having compact essential support and satisfying

$$\lim_{k\to\infty} \int_0^\infty f_k(s)\phi(s)\,\mathrm{d}s = \phi(t) \tag{3.5}$$

for every complex-valued, continuous function ϕ on $\mathbb{R}^+$. Given $n \in \mathbb{N}$, taking $\phi(s) = e^{-\mathrm{i}\omega_n s}$ in (3.5) yields

$$\lim_{k\to\infty} \mathscr{F}_{\omega_n}(f_k) = \mathrm{e}^{-\mathrm{i}\omega_n t}.$$

Since, by definition,

$$(H_\omega(f)x)_n = \mathscr{F}_{\omega_n}(f)x_n$$

for each $f \in L^1(\mathbb{R}^+)$, it follows that

$$\lim_{k\to\infty} (H_\omega(f_k)x)_n = \lim_{k\to\infty} \mathscr{F}_{\omega_n}(f_k)x_n = \mathrm{e}^{-\mathrm{i}\omega_n t} x_n. \tag{3.6}$$

On the other hand,

$$(H_\omega(f)x)_n = \int_0^\infty f(s)(h_{x,\omega}(s))_n\,\mathrm{d}s$$

for each $f \in L^1(\mathbb{R}^+)$, so substituting $\phi(s) = (h_{x,\omega}(s))_n$ in (3.5) gives

$$\lim_{k\to\infty} (H_\omega(f_k)x)_n = (h_{x,\omega}(t))_n.$$

Comparing this with (3.6), we see that

$$(h_{x,\omega}(t))_n = \mathrm{e}^{-\mathrm{i}\omega_n t} x_n = (\mathscr{V}_\omega(t)x)_n.$$

Since t and n are arbitrary, we conclude that $h_{x,\omega}(t) = \mathscr{V}_\omega(t)x$ for each $t \in \mathbb{R}^+$. □

Proof of Theorem 3.4. Assume towards a contradiction that H_ω is the integrated form of a uniformly bounded, strongly continuous semigroup $\mathscr{S} = \{\mathscr{S}(t)\}_{t\in\mathbb{R}^+}$ on ℓ^∞. The family $\mathscr{V}_\omega = \{\mathscr{V}_\omega(t)\}_{t\in\mathbb{R}^+}$ forms a one-parameter semigroup on ℓ^∞. By Lemma 3.6, $\mathscr{S}$ coincides with $\mathscr{V}_\omega$. However, Lemma 3.5 shows that $\mathscr{V}_\omega$ is not strongly continuous. This contradicts the strong continuity of $\mathscr{S}$, and completes the proof. □

4. Associated representations

This section introduces representations derived from algebra homomorphisms, which we call *associated representations.* Given an algebra homomorphism, the associated representation is a naturally defined representation of the domain algebra that acts on the range space of the homomorphism. We shall specifically explore associated representations arising from homomorphisms of semigroup algebras. These will play an important role in the sections that follow.

Let $\mathfrak{A}$ be a normed algebra. For each $x \in \mathfrak{A}$, let L_x denote the operator in $\mathscr{L}(\mathfrak{A})$ defined by

$$L_x y = xy \quad (y \in \mathfrak{A}).$$

Then the mapping

$$L\colon \mathfrak{A} \to \mathscr{L}(\mathfrak{A}), \quad x \mapsto L_x,$$

is a continuous homomorphism from $\mathfrak{A}$ into $\mathscr{L}(\mathfrak{A})$. It is standard to refer to L as the *left regular representation* of $\mathfrak{A}$ on itself. If $\mathfrak{A}$ is commutative, then L is simply called the regular representation.

Given a mapping $f\colon A \to B$, where A and B are sets, we denote by $\operatorname{Ran}(f)$ the range of f.

Let X be a normed space. Given $S \in \mathscr{L}(X)$ and a linear subspace $Y \subset X$ such that $S(Y) \subset Y$, we denote by $S{\restriction_Y}$ the restriction of S to Y, that is, the linear operator in $\mathscr{L}(Y)$ defined by

$$(S{\restriction_Y})y = Sy \quad (y \in Y).$$

Given a mapping $f\colon A \to \mathscr{L}(X)$, where A is a set, and a linear subspace $Y \subset X$ such that $(f(a))(Y) \subset Y$ for each $a \in A$, we denote by $f{\restriction_Y}$ the mapping

$$f{\restriction_Y}\colon A \to \mathscr{L}(Y), \quad a \mapsto f(a){\restriction_Y}.$$

Let $\mathfrak{X}$ and $\mathfrak{A}$ be normed algebras, and let H be a homomorphism from $\mathfrak{X}$ into $\mathfrak{A}$. Continuing to denote by L the left regular representation of $\mathfrak{A}$ on itself, we have

$$\big((L\circ H)(x)\big)H(y) = L_{H(x)}H(y) = H(x)H(y) = H(xy)$$

for all $x, y \in \mathfrak{X}$. Hence

$$\big((L\circ H)(x)\big)(\operatorname{Ran}(H)) \subset \operatorname{Ran}(H)$$

for each $x \in \mathfrak{X}$, and we may therefore consider the mapping

$$(L\circ H){\restriction_{\operatorname{Ran}(H)}}\colon \mathfrak{X} \to \mathscr{L}(\operatorname{Ran}(H)), \quad x \mapsto (L\circ H){\restriction_{\operatorname{Ran}(H)}}(x).$$

It is immediate that

$$\big((L\circ H){\restriction_{\operatorname{Ran}(H)}}\big)(x)z = H(x)z$$

for each $x \in \mathfrak{X}$ and each $z \in \operatorname{Ran}(H)$. It is also clear that $(L \circ H)\restriction_{\operatorname{Ran}(H)}$ is a representation of $\mathfrak{A}$ on $\operatorname{Ran}(H)$. If H is continuous, then $(L\circ H)\restriction_{\operatorname{Ran}(H)}$ is continuous as well. We shall refer to $(L \circ H)\restriction_{\operatorname{Ran}(H)}$ as the *representation of* $\mathfrak{X}$ *on* $\operatorname{Ran}(H)$ *associated with* H.

We are now ready to present the main (and only) result of this section. This result generalises [5, Proposition 3.4] and is inspired by Kisyński's work on the algebraic version of the Hille–Yosida theorem (cf. [24, Theorems 4.2 and 5.5], [25, Theorems 10.2 and 12.5], and also [8, Theorems 3.3 and 4.1]).

Proposition 4.1. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G, and let $\mathfrak{A}$ be a normed algebra. Let $H\colon L^1(S) \to \mathfrak{A}$ be a non-zero, continuous homomorphism. Then:*

(i) *there exists a unique semigroup $\mathscr{S} = \{\mathscr{S}(s)\}_{s\in S_D}$ on $\operatorname{Ran}(H)$ such that*

$$\mathscr{S}(s)H(f) = H(S_s f) \tag{4.1}$$

for each $s \in S_D$ and each $f \in L^1(S)$; the semigroup $\mathscr{S}$ is strongly continuous and satisfies $\sup_{s\in S_D} \|\mathscr{S}(s)\| \le \|H\|$;

(ii) *if $s \in S_D$ and if $\{f_\iota\}_{\iota\in I}$ is as defined in Definition 2.1, then*

$$\mathscr{S}(s)x = \lim_{\iota\in I} H(f_\iota)x \tag{4.2}$$

for each $x \in \operatorname{Ran}(H)$;

(iii) *the representation of $L^1(S)$ on $\operatorname{Ran}(H)$ associated with H is the integrated form of $\mathscr{S}$:*

$$\big((L \circ H)\restriction_{\operatorname{Ran}(H)}\big)(f)x =: H(f)x = \int_{S_D} f(s)\mathscr{S}(s)x \,\mathrm{d}m_G(s) \tag{4.3}$$

for each $f \in L^1(S)$ and each $x \in \operatorname{Ran}(H)$, where L denotes the left regular representation of $\mathfrak{A}$ on itself.

Proof. (i) First, note that since H is non-zero, so too is $\operatorname{Ran}(H)$.

Next, let $f, g \in L^1(S)$. The function $S \ni s \mapsto S_s g \in L^1(S)$ is continuous, so the function $S \ni s \mapsto f(s)S_s g \in L^1(S)$ is Haar measurable. Since the function $S \ni s \mapsto f(s)S_s g \in L^1(S)$ is also dominated in norm by the Haar integrable function $s \mapsto |f(s)|\|g\|_1$, it is Bochner integrable. One verifies at once that

$$\int_S f(s)S_s g \,\mathrm{d}m_G(s) = f \star g, \tag{4.4}$$

where the integral is understood as a Bochner integral. We shall use this formula shortly.

Let $\mathfrak{B}$ be the Banach algebra completion of $\operatorname{Ran}(H)$. The continuity of the function $S \ni s \mapsto S_s g \in L^1(S)$ and the continuity of H imply that the function $S \ni s \mapsto H(S_s g) \in \mathfrak{B}$ is continuous. The function $S \ni s \mapsto f(s)H(S_s g) \in \mathfrak{B}$ is therefore Haar measurable, and since it is also dominated in norm by the Haar integrable function $s \mapsto \|H\||f(s)|\|g\|_1$, it is Bochner integrable. It turns out that

$$\int_S f(s)H(S_s g) \,\mathrm{d}m_G(s) = H(f)H(g), \tag{4.5}$$

where the integral is understood in the Bochner sense. Indeed, by the interchangeability of the Bochner integral with bounded linear operators and by (4.4), we have

$$\int_S f(s)H(S_s g) \,\mathrm{d}m_G(s) = H\bigg(\int_S f(s)S_s g \,\mathrm{d}m_G(s)\bigg) = H(f \star g) = H(f)H(g).$$

Note that the leftmost term above is a priori a Bochner integral with values in $\mathfrak{B}$, but it is actually a member of $\operatorname{Ran}(H)$, as seen from the middle equality.

Given $s \in S_D$, we define a linear operator $\mathscr{S}(s)$ on $\mathrm{Ran}(H)$ by

$$\mathscr{S}(s)H(f) = H(S_s f) \quad (f \in L^1(S)).$$

We shall verify shortly that this operator is well defined. Once this has been established, the linearity of $\mathscr{S}(s)$ follows immediately from the linearity of H and of the mapping $f \mapsto S_s f$.

To establish the well-definedness of $\mathscr{S}(s)$, we need to show that, if $H(g) = H(h)$ for $g, h \in L^1(S)$, then

$$H(S_s g) = H(S_s h). \tag{4.6}$$

Let $f \in L^1(S)$. Then, in view of (4.5), we have

$$\int_S f(t)H(S_t g)\,\mathrm{d}m_G(t) = H(f)H(g) = H(f)H(h) = \int_S f(t)H(S_t h)\,\mathrm{d}m_G(t).$$

The equality of the leftmost and rightmost integrals, combined with the arbitrariness of $f \in L^1(S)$, implies that the functions $t \mapsto H(S_t g)$ and $t \mapsto H(S_t h)$ are equal locally almost everywhere on S. But, since both of these functions are continuous, Lemma 2.2 ensures they are in fact identical on S_D, which establishes (4.6) for our chosen $s \in S_D$.

Using the semigroup property of $\{S_s\}_{s\in S}$, one verifies at once that $\{\mathscr{S}(s)\}_{s\in S_D}$ is a semigroup on $\mathrm{Ran}(H)$. The strong continuity of $\mathscr{S}$ follows immediately from the strong continuity of $\{S_s\}_{s\in S}$, and the uniqueness of $\mathscr{S}$ follows from the defining property given in (4.1).

Let $s \in S_D$, let $\{f_\iota\}_{\iota\in I}$ be as defined in Definition 2.1, and let $f \in L^1(S)$. Since the function $t \mapsto H(S_t f)$ is continuous, it follows by a standard argument that

$$\lim_{\iota\in I} \int_S f_\iota(t)H(S_t f)\,\mathrm{d}m_G(t) = H(S_s f). \tag{4.7}$$

In view of (4.5), for each $\iota \in I$, we have

$$\int_S f_\iota(t)H(S_t f)\,\mathrm{d}m_G(t) = H(f_\iota)H(f).$$

Therefore, (4.7) may be restated as

$$\lim_{\iota\in I} H(f_\iota)H(f) = H(S_s f). \tag{4.8}$$

For each $\iota \in I$, we have

$$\|H(f_\iota)H(f)\| \le \|H(f_\iota)\|\,\|H(f)\| \le \|H\|\,\|f_\iota\|_1\|H(f)\| = \|H\|\,\|H(f)\|.$$

Hence, by (4.8),

$$\|H(S_s f)\| \le \|H\|\,\|H(f)\|,$$

which implies that $\mathscr{S}(s)$ is bounded, with norm at most $\|H\|$. Since s was arbitrarily chosen from S_D, we conclude that $\sup_{s\in S_D} \|\mathscr{S}(s)\| \le \|H\|$.

(ii) Formula (4.2) is a restatement of (4.8).

(iii) Let $f, g \in L^1(S)$. Then, since

$$\mathscr{S}(s)H(g) = H(S_s g) \quad (s \in S_D),$$

(4.5) can be rewritten as

$$H(f)H(g) = \int_S f(s)H(S_s g)\,\mathrm{d}m_G(s) = \int_{S_D} f(s)\mathscr{S}(s)H(g)\,\mathrm{d}m_G(s).$$

This gives (4.3) if we set $x = H(g)$. □

5. Characters on semigroup algebras

In this section, we characterise the characters on semigroup algebras within the class of algebras relevant to this work. This characterisation plays only a supporting role: its purpose is to place our main result—established in the following section—in the proper context. Although similar results exist in the literature, the characterisation below appears to be the first explicit formulation for the class of semigroup algebras that we consider here. Our approach draws on the associated representations introduced in the previous section and follows a strategy akin to that employed in [32], where a theorem describing the form of non-degenerate Hilbert space $*$-representations of group algebras is used to determine the characters on group algebras of locally compact Abelian groups (see the proof of [32, Corollary 10.1.1]).

Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. We denote by $\widehat{S}^\star$ the set of all continuous semi-characters on S that are not locally almost everywhere zero. In general, $\widehat{S}^\star$ is a proper subset of $\widehat{S}$ (the set of all continuous semi-characters on S). For example, if $G = \mathbb{R}$ and $S = \{0\} \cup [1, \infty)$, then the function $\zeta \colon S \to \mathbb{R}$ defined by

$$\zeta(s) = \begin{cases} 1 & \text{if } s = 0, \\ 0 & \text{if } s \in [1, \infty) \end{cases}$$

is a continuous semi-character on S that is zero almost everywhere on S, and thus lies in $\widehat{S} \setminus \widehat{S}^\star$. When $S \subset D(S)$, we have $\widehat{S}^\star = \widehat{S}$. Indeed, in this case $S = S_D$, so by Lemma 2.2, any continuous function on S that vanishes locally almost everywhere is identically zero, whereas semi-characters, by definition, are not identically zero. One condition ensuring $S \subset D(S)$ is that the identity element e_G of G belongs to $D(S)$. This condition holds, for example, when $G = \mathbb{R}$, equipped with either the usual or discrete topology, and $S = \mathbb{R}^+$. It also applies when $G = \mathbb{Z}$ and $S = \mathbb{Z}^+$.

For each $f \in L^1(S)$, the *generalised Laplace transform* of f is the function $\zeta \mapsto \mathscr{L}_\zeta(f)$ on $\widehat{S}^\star$, defined by

$$\mathscr{L}_\zeta(f) = \int_S f(s)\zeta(s)\,\mathrm{d}m_G(s) \quad (\zeta \in \widehat{S}^\star).$$

For each $\zeta \in \widehat{S}^\star$, the map

$$\mathscr{L}_\zeta \colon L^1(S) \to \mathbb{C}, \quad f \mapsto \mathscr{L}_\zeta(f),$$

is a continuous linear functional on $L^1(S)$, with norm $\operatorname{ess\,sup}_{s\in S} |\zeta(s)| \le 1$. As we shall see shortly, $\mathscr{L}_\zeta$ is multiplicative, and hence a character on $L^1(S)$.

The following theorem makes this precise by identifying the character space $\Delta(L^1(S))$ with the set $\{\mathscr{L}_\zeta : \zeta \in \widehat{S}^\star\}$. This result extends earlier characterisations of this kind established for commutative semigroup algebras (see [13, Theorem 5.1] and [29, Theorem 13]), for commutative group algebras (see [12, Theorem 4.5.4] and [30, Theorem 1.2.2]), and for general group algebras (see [20, Theorem 23.4 and Corollary 23.7]).

Theorem 5.1. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Then:*

(i) *for each $\zeta \in \widehat{S}^\star$, the functional $\mathscr{L}_\zeta$ is a character on $L^1(S)$;*
(ii) *every character on $L^1(S)$ is of the form $\mathscr{L}_\zeta$ for a unique $\zeta \in \widehat{S}^\star$.*

A lemma will precede the proof of this theorem.

Lemma 5.2. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. If two semi-characters in $\widehat{S}^\star$ coincide locally almost everywhere, then they coincide everywhere.*

Proof. Suppose that $\zeta_1, \zeta_2 \in \widehat{S}^\star$ are such that $\zeta_1 = \zeta_2$ locally almost everywhere on S. Then $\mathscr{L}_{\zeta_1} = \mathscr{L}_{\zeta_2}$. Set $\phi = \mathscr{L}_{\zeta_1}$ $(= \mathscr{L}_{\zeta_2})$. Since ζ_1 is not zero locally almost everywhere, it follows that ϕ is non-zero. Choose $g_0 \in L^1(S)$ such that $\phi(g_0) \neq 0$. Let $s \in S$. Then, for $i = 1, 2$, we have

$$\begin{aligned}\phi(S_s g_0) &= \int_S S_s g_0(t)\zeta_i(t)\,\mathrm{d}m_G(t) = \int_{sS} g_0(s^{-1}t)\zeta_i(t)\,\mathrm{d}m_G(t)\\ &= \zeta_i(s)\int_{sS} g_0(s^{-1}t)\zeta_i(s^{-1}t)\,\mathrm{d}m_G(t)\\ &= \zeta_i(s)\int_S g_0(t)\zeta_i(t)\,\mathrm{d}m_G(t) = \zeta_i(s)\phi(g_0).\end{aligned}$$

This implies that

$$\zeta_i(s) = \frac{\phi(S_s g_0)}{\phi(g_0)}$$

for $i = 1, 2$, and hence $\zeta_1(s) = \zeta_2(s)$. □

Proof of Theorem 5.1. (i) Let $\zeta \in \widehat{S}^\star$. Since ζ is not zero locally almost everywhere, $\mathscr{L}_\zeta$ is non-zero. Therefore, to show that $\mathscr{L}_\zeta$ is a character on $L^1(S)$, we only need to show that $\mathscr{L}_\zeta$ is multiplicative.

First, note that the regular representation of $\mathbb{C}$, $\mathbb{C} \ni a \mapsto L_a \in \mathscr{L}(\mathbb{C})$, is a topological isomorphism of algebras. For each $s \in S$, set

$$\mathscr{S}(s) = L_{\zeta(s)}.$$

Clearly, $\{\mathscr{S}(s)\}_{s\in S}$ is a uniformly bounded, strongly continuous semigroup on $\mathbb{C}$, and so, *a fortiori*, is $\{\mathscr{S}(s)\}_{s\in S_D}$. It is also straightforward to see that the integrated form of $\{\mathscr{S}(s)\}_{s\in S_D}$ is the map

$$H\colon L^1(S) \to \mathscr{L}(\mathbb{C}), \quad f \mapsto H(f) := L_{\mathscr{L}_\zeta(f)}.$$

By Theorem 3.2, H is a homomorphism of algebras. Now, for any $f, g \in L^1(S)$, the relation

$$H(f)H(g) = H(f \star g),$$

which can be written as

$$L_{\mathscr{L}_\zeta(f)} L_{\mathscr{L}_\zeta(g)} = L_{\mathscr{L}_\zeta(f\star g)},$$

together with the identity

$$L_{\mathscr{L}_\zeta(f)} L_{\mathscr{L}_\zeta(g)} = L_{\mathscr{L}_\zeta(f)\mathscr{L}_\zeta(g)},$$

which captures the homomorphic property of L, yields

$$\mathscr{L}_\zeta(f)\mathscr{L}_\zeta(g) = \mathscr{L}_\zeta(f \star g),$$

as required.

(ii) Suppose that $\phi \in \Delta(L^1(S))$, so that $\phi\colon L^1(S) \to \mathbb{C}$ is a non-zero, continuous homomorphism. By Proposition 4.1, there exists a bounded, strongly continuous semigroup $\mathscr{S} = \{\mathscr{S}(s)\}_{s\in S_D}$ on $\mathrm{Ran}(\phi)$, determined by ϕ, that satisfies the properties stated in that proposition. Since $\mathbb{C}$ has no proper, non-zero, complex linear subspaces, we have $\mathrm{Ran}(\phi) = \mathbb{C}$. Moreover, each element of $\mathscr{L}(\mathbb{C})$ takes the form L_a for some unique $a \in \mathbb{C}$. Therefore, for each $s \in S_D$, there exists a unique $\zeta_0(s) \in \mathbb{C}$ such that $\mathscr{S}(s) = L_{\zeta_0(s)}$. For any $s, t \in S_D$, the relation

$$\mathscr{S}(s)\mathscr{S}(t) = \mathscr{S}(st)$$

written as
$$L_{\zeta_0(s)}L_{\zeta_0(t)} = L_{\zeta_0(st)}$$
and combined with
$$L_{\zeta_0(s)}L_{\zeta_0(t)} = L_{\zeta_0(s)\zeta_0(t)}$$
implies that
$$\zeta_0(s)\zeta_0(t) = \zeta_0(st).$$
Since $\mathscr{S}$ is strongly continuous and since
$$\mathscr{S}(s)1 = L_{\zeta_0(s)}1 = \zeta_0(s) \quad (s \in S_D),$$
the function $\zeta_0 \colon S_D \to \mathbb{C}$ is continuous. Moreover, for each $s \in S_D$, we have
$$|\zeta_0(s)| = \|L_{\zeta_0(s)}\| = \|\mathscr{S}(s)\| \leq \|\phi\| \leq 1,$$
so ζ_0 assumes values in $\overline{\mathbb{D}}$. Therefore, ζ_0 is a semi-character in $\widehat{S_D}$.

Continuing to exploit Proposition 4.1, we now state two critical formulae involving ζ_0. First, if $s \in S_D$ and $f \in L^1(S)$, then
$$\mathscr{S}(s)\phi(f) = L_{\zeta_0(s)}\phi(f) = \zeta_0(s)\phi(f),$$
and so we can rewrite (4.1) as
$$\zeta_0(s)\phi(f) = \phi(S_s f) \quad (s \in S_D,\ f \in L^1(S)). \tag{5.1}$$
Second, rewriting (4.3) as
$$\phi(f)x = \int_{S_D} f(s)\zeta_0(s)x \,\mathrm{d}m_G(s) \quad (f \in L^1(S),\ x \in \mathbb{C})$$
and letting $x = 1$, we obtain
$$\phi(f) = \int_{S_D} f(s)\zeta_0(s) \,\mathrm{d}m_G(s) \quad (f \in L^1(S)). \tag{5.2}$$

Next, we shall show that ζ_0 extends to a semi-character in $\widehat{S}^{\star}$.

Select $g_0 \in L^1(S)$ such that $\phi(g_0) \neq 0$, and define a function $\zeta \colon S \to \mathbb{C}$ by
$$\zeta(s) = \frac{\phi(S_s g_0)}{\phi(g_0)} \quad (s \in S). \tag{5.3}$$
When we replace f with g_0 in (5.1) and compare the result with (5.3), we see that ζ_0 is equal to the restriction of ζ to S_D. We shall show that ζ is a semi-character in $\widehat{S}$.

First, observe that ζ is continuous, since the function $S \ni t \mapsto \phi(S_t g_0) \in \mathbb{C}$ is continuous, being the composition of the continuous maps $S \ni t \mapsto S_t g_0 \in L^1(S)$ and ϕ.

Next, note that ζ is bounded in modulus by $\|g_0\|_1/|\phi(g_0)|$.

Further, since ζ_0 and ζ agree on S_D, we can restate (5.1) as
$$\zeta(s)\phi(f) = \phi(S_s f) \quad (s \in S_D,\ f \in L^1(S)). \tag{5.4}$$
Based on this, we shall now demonstrate that
$$\zeta(st) = \zeta(s)\zeta(t) \quad (s \in S_D,\ t \in S). \tag{5.5}$$

Let $s \in S_D$ and $t \in S$. By applying (5.4) with f replaced by $S_t g_0$, we obtain
$$\zeta(s)\phi(S_t g_0) = \phi(S_s(S_t g_0)) = \phi(S_{st} g_0). \tag{5.6}$$
Applying (5.3) with s replaced by t yields
$$\phi(S_t g_0) = \zeta(t)\phi(g_0).$$
In turn, applying (5.3) with s replaced by st gives
$$\phi(S_{st} g_0) = \zeta(st)\phi(g_0).$$

Thus, we can rewrite (5.6) as

$$\zeta(s)\zeta(t)\phi(g_0) = \zeta(st)\phi(g_0),$$

and this, given that $\phi(g_0) \neq 0$, immediately yields (5.5).

We shall now show that (5.5) can be further generalised to the form

$$\zeta(t_1t_2) = \zeta(t_1)\zeta(t_2) \quad (t_1, t_2 \in S). \tag{5.7}$$

Choose $s \in S_D$ such that $\zeta(s) \neq 0$, which is possible since ζ is not zero locally almost everywhere, and take $t_1, t_2 \in S$. Applying (5.5) with $t = t_1t_2$, we find that

$$\zeta(st_1t_2) = \zeta(s)\zeta(t_1t_2). \tag{5.8}$$

Since S_D is an ideal in S, st_1 belongs to S_D along with s, and we can apply (5.5) with s replaced by st_1 and with t replaced by t_2 to yield

$$\zeta(st_1t_2) = \zeta(st_1)\zeta(t_2). \tag{5.9}$$

Applying (5.5) with $t = t_1$, we also obtain

$$\zeta(st_1) = \zeta(s)\zeta(t_1).$$

This allows us to rewrite (5.9) as

$$\zeta(st_1t_2) = \zeta(s)\zeta(t_1)\zeta(t_2).$$

Comparing the above formula with (5.8), we see that

$$\zeta(s)\zeta(t_1t_2) = \zeta(s)\zeta(t_1)\zeta(t_2).$$

Since $\zeta(s) \neq 0$, we conclude that (5.7) holds.

Thus, ζ has all the defining properties of a semi-character in $\widehat{S}$.

Since ζ_0 and ζ agree on S_D and since S_D is locally of full measure in S, we can rewrite (5.2) as

$$\phi(f) = \int_S f(s)\zeta(s)\,\mathrm{d}m_G(s) =: \mathscr{L}_\zeta(f) \quad (f \in L^1(S)),$$

which is the desired representation of ϕ.

Finally, the uniqueness of the representation of ϕ in question follows immediately from Lemma 5.2. □

We now state a corollary of the theorem just proved.

Corollary 5.3. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Then every semi-character in $\widehat{S_D}$ extends uniquely to a semi-character in $\widehat{S}^\star$.*

Proof. Let $\zeta \in \widehat{S_D}$. Then ζ is not locally almost everywhere zero on S_D, for otherwise, by Lemma 2.2, it would be identically zero, contradicting the definition of a semi-character. Consequently, by Theorem 5.1(i), ζ gives rise to the character $\mathscr{L}_\zeta$ on $L^1(S_D)$. Since $L^1(S_D) = L^1(S)$, $\mathscr{L}_\zeta$ can equally well be regarded as a character on $L^1(S)$. By Theorem 5.1(ii), considered as a character on $L^1(S)$, it must be of the form $\mathscr{L}_\chi$ for some $\chi \in \widehat{S}^\star$.

Now, $\mathscr{L}_\zeta = \mathscr{L}_\chi$ implies that χ and ζ coincide locally almost everywhere on S_D. Since both χ and ζ are continuous, Lemma 2.2 ensures that they coincide on all of S_D. Thus, χ is an extension of ζ.

If $\tilde{\chi} \in \widehat{S}^\star$ is another extension of ζ, then $\tilde{\chi}$ and χ, both agreeing with ζ on S_D, coincide on S_D, and hence locally almost everywhere on S. Lemma 5.2 then ensures that $\tilde{\chi}$ and χ coincide on all of S. □

Note that, in the setting of Corollary 5.3, the restriction of a semi-character in $\widehat{S}^\star$ to the subsemigroup S_D is always a semi-character in $\widehat{S_D}$. This follows from the simple observation that a function that is not locally almost everywhere zero on S cannot be locally almost everywhere zero on S_D. Combining Corollary 5.3 with this observation, and adopting the notation $\zeta|_{S_D}$ for the restriction of $\zeta \in \widehat{S}^\star$ to S_D, we arrive at the following result.

Proposition 5.4. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Then the restriction map*

$$\widehat{S}^\star \to \widehat{S_D}, \quad \zeta \mapsto \zeta|_{S_D},$$

is a bijection.

Note that, in the setting of Proposition 5.4 (equivalently, of Theorem 5.1), we have $\mathscr{L}_\zeta = \mathscr{L}_{\zeta|_{S_D}}$ for each $\zeta \in \widehat{S}^\star$. Combining this fact with Proposition 5.4 and Theorem 5.1, we see that the character space $\Delta(L^1(S))$ admits two equivalent descriptions: as the image of $\widehat{S}^\star$ under the mapping $\zeta \mapsto \mathscr{L}_\zeta$, and as the image of $\widehat{S_D}$ under the same mapping. We record this as the following extension of Theorem 5.1.

Theorem 5.5. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Let $\Phi \in \{\widehat{S}^\star, \widehat{S_D}\}$. Then, for each $\zeta \in \Phi$, $\mathscr{L}_\zeta \in \Delta(L^1(S))$, and the map $\zeta \mapsto \mathscr{L}_\zeta$ is a bijection from Φ onto $\Delta(L^1(S))$. Consequently,*

$$\Delta(L^1(S)) = \{\mathscr{L}_\zeta : \zeta \in \widehat{S}^\star\} = \{\mathscr{L}_\zeta : \zeta \in \widehat{S_D}\}.$$

6. Isolability of homomorphisms

In this section, we establish our main result concerning the isolability of homomorphisms from relevant semigroup algebras. As in the previous section, our approach will build upon associated representations introduced earlier.

We begin by presenting the necessary ingredients for our analysis, starting with an elementary result.

Proposition 6.1. *Let $\mathfrak{A}$ be a unital normed algebra, and let $p \in \mathfrak{A}$ be an idempotent. If*

$$\|p - e_{\mathfrak{A}}\| < 1,$$

then $p = e_{\mathfrak{A}}$.

Proof. Let $\mathfrak{B}$ be the Banach algebra completion of $\mathfrak{A}$. Define an element r in $\mathfrak{B}$ by

$$r = e_{\mathfrak{A}} + \sum_{n=1}^{\infty} (e_{\mathfrak{A}} - p)^n.$$

Since $\|e_{\mathfrak{A}} - p\| < 1$, the series converges absolutely in $\mathfrak{B}$. Using the identity

$$(e_{\mathfrak{A}} - x)\left(e_{\mathfrak{A}} + \sum_{n=1}^{\infty} x^n\right) = e_{\mathfrak{A}} \quad (x \in \mathfrak{B},\ \|x\| < 1)$$

with x taken to be $e_{\mathfrak{A}} - p$, we see that $pr = e_{\mathfrak{A}}$. But then

$$p^2 r = p e_{\mathfrak{A}} = p$$

and also, since p is idempotent,

$$p^2 r = pr = e_{\mathfrak{A}}.$$

Hence $p = e_{\mathfrak{A}}$, as required. □

Another ingredient is the following result, which is known but perhaps not widely so.

Proposition 6.2. *Let $\mathfrak{A}$ be a unital normed algebra, and let $x \in \mathfrak{A}$. If*

$$\sup_{n\in\mathbb{N}} \|x^n - e_{\mathfrak{A}}\| < 1,$$

then $x = e_{\mathfrak{A}}$.

This result has been attributed to several authors. Cox [11] first established it for square matrices. Nakamura and Yoshida [27] extended Cox's result to bounded linear operators on a Hilbert space, and Hirschfeld [22] further extended it to elements of a normed algebra. Wallen [34] later provided a concise and enlightening proof of a slightly more general version of Hirschfeld's result. Further related contributions were made by Wils [36], Chernoff [7], Nagisa and Wada [26], and Kalton et al. [23].

It is worth noting that Proposition 6.2 directly implies Proposition 6.1.

Alternative proof of Proposition 6.1. The idempotency of p implies that $\|p^n - e_{\mathfrak{A}}\| = \|p - e_{\mathfrak{A}}\|$ for each $n \in \mathbb{N}$. It follows that $\sup_{n\in\mathbb{N}} \|p^n - e_{\mathfrak{A}}\| < 1$. An application of Proposition 6.2 now establishes that $p = e_{\mathfrak{A}}$. $\square$

We require one more, and final, ingredient before we proceed.

Given a Haar measurable subset A of a locally compact group, we denote by $\mathcal{M}^{\mathrm{fin}}(A)$ the collection of all Haar measurable subsets of A with finite Haar measure.

Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Let $\sigma \in \widehat{S}_{\mathrm{u}}$, let $r > 0$, and let $s \in S$. We denote by $\mathcal{A}_{s,\sigma,r}(S)$ the collection of all sets A in $\mathcal{M}^{\mathrm{fin}}(S)$ such that:

(i) $0 < m_G(A)$;

(ii) $|\sigma(t) - \sigma(s)| \leq r$ for all $t \in A$.

Lemma 6.3. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Let $\sigma \in \widehat{S}_{\mathrm{u}}$, and let $r > 0$. Then:*

(i) *for each $s \in S_D$, the family $\mathcal{A}_{s,\sigma,r}(S)$ is non-empty;*

(ii) *the set*

$$\{\chi_A : A \in \mathcal{A}_{s,\sigma,r}(S),\ s \in S_D\}$$

is linearly dense in $L^1(S)$ with respect to the L^1-norm.

Proof. To prove (i), let $s \in S_D$ and let $U \subset G$ be an open set containing s. Since σ is continuous and every element of G has a local base of precompact open sets, there exists a precompact open neighbourhood $V \subset G$ of s such that $|\sigma(t) - \sigma(s)| \leq r$ for all $t \in V \cap S$. Define

$$W = V \cap U.$$

As $s \in S_D$ and W is an open neighbourhood of s, we have $m_G(W \cap S) > 0$. Since W is precompact, we also have $m_G(W \cap S) < \infty$. Furthermore, as $W \cap S \subset V \cap S$, we have $|\sigma(t) - \sigma(s)| \leq r$ for all $t \in W \cap S$. Therefore, $W \cap S \in \mathcal{A}_{s,\sigma,r}(S)$, which completes the proof of (i).

To prove (ii), since simple functions supported on sets in $\mathcal{M}^{\mathrm{fin}}(S)$ are linearly dense in $L^1(S)$ with respect to the L^1-norm, we need only show that, for each $E \in \mathcal{M}^{\mathrm{fin}}(S_D)$, the characteristic function χ_E can be approximated in the L^1-norm by finite linear combinations of characteristic functions χ_A, where $A \in \mathcal{A}_{s,\sigma,r}(S)$ and $s \in S_D$. Since $S \setminus S_D$ is locally null, $\chi_E = \chi_{E\cap S_D}$ almost everywhere. Therefore, without loss of generality, we may assume that $E \subset S_D$.

Let $E \in \mathcal{M}^{\mathrm{fin}}(S_D)$, and fix $\epsilon > 0$. By the inner regularity of Haar measure for finite-measure sets, there exists a compact set $K \subset E$ with $m_G(E) - m_G(K) < \epsilon$. By the outer regularity of Haar measure for all measurable sets, there exists an open set $U \supset E$ with $m_G(U) - m_G(E) < \epsilon$. By the argument from the proof of (i), and since $K \subset S_D$, for each $s \in K$, there exists an open set $W_s \subset U$ containing s such that $Z_s = W_s \cap S \in \mathcal{A}_{s,\sigma,r}(S)$. As K is compact, there exists a finite set

$\Sigma = \{s_1, \ldots, s_I\} \subset K$ such that $\{W_s\}_{s\in\Sigma}$—and hence also $\{Z_s\}_{s\in\Sigma}$—covers K. Define the sets A_s for $s \in \Sigma$ as follows:

$$A_{s_1} = Z_{s_1}, \quad A_{s_i} = Z_{s_i} \setminus \bigcup_{j=1}^{i-1} Z_{s_j} \quad \text{for } i = 2, \ldots, I.$$

The sets A_s, $s \in \Sigma$, are disjoint, Haar measurable, and satisfy $A_s \subset Z_s$ for each $s \in \Sigma$. Their union coincides with $\bigcup_{s\in\Sigma} Z_s$, which has positive Haar measure since each Z_s with $s \in K$ does. Hence the set

$$\Sigma' = \{s \in \Sigma : m_G(A_s) > 0\}$$

is non-empty. For every $s \in S$, if $A \subset B \in \mathcal{A}_{s,\sigma,r}(S)$ is Haar measurable and $m_G(A) > 0$, then $A \in \mathcal{A}_{s,\sigma,r}(S)$. Thus, as $A_s \subset Z_s \in \mathcal{A}_{s,\sigma,r}(S)$ for each $s \in \Sigma$, and $m_G(A_s) > 0$ for each $s \in \Sigma'$, it follows that $A_s \in \mathcal{A}_{s,\sigma,r}(S)$ for every $s \in \Sigma'$.

Since the sets A_s, $s \in \Sigma$, are disjoint, we have

$$m_G\left(\bigcup_{s\in\Sigma'} A_s\right) = \sum_{s\in\Sigma'} m_G(A_s) = \sum_{s\in\Sigma} m_G(A_s) = m_G\left(\bigcup_{s\in\Sigma} Z_s\right).$$

Hence, as $\bigcup_{s\in\Sigma'} A_s \subset \bigcup_{s\in\Sigma} Z_s$, it follows that

$$\left\|\chi_{\bigcup_{s\in\Sigma} Z_s} - \chi_{\bigcup_{s\in\Sigma'} A_s}\right\|_1 = m_G\left(\bigcup_{s\in\Sigma} Z_s\right) - m_G\left(\bigcup_{s\in\Sigma'} A_s\right) = 0,$$

implying that $\chi_{\bigcup_{s\in\Sigma} Z_s}$ and $\chi_{\bigcup_{s\in\Sigma'} A_s}$ determine the same element of $L^1(S)$.

Since $K \subset \bigcup_{s\in\Sigma} Z_s$, we have

$$\left\|\chi_{\bigcup_{s\in\Sigma} Z_s} - \chi_K\right\|_1 = m_G\left(\bigcup_{s\in\Sigma} Z_s\right) - m_G(K).$$

Furthermore, as $\bigcup_{s\in\Sigma} Z_s \subset U$,

$$\begin{aligned} m_G\left(\bigcup_{s\in\Sigma} Z_s\right) - m_G(K) &\le m_G(U) - m_G(K) \\ &= [m_G(U) - m_G(E)] + [m_G(E) - m_G(K)] \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Hence

$$\left\|\chi_{\bigcup_{s\in\Sigma} Z_s} - \chi_K\right\|_1 \le 2\epsilon.$$

Since $K \subset E$, we also have

$$\|\chi_E - \chi_K\|_1 = m_G(E) - m_G(K) < \epsilon.$$

Consequently,

$$\begin{aligned} \left\|\chi_E - \sum_{s\in\Sigma'} \chi_{A_s}\right\|_1 &= \left\|\chi_E - \chi_{\bigcup_{s\in\Sigma'} A_s}\right\|_1 = \left\|\chi_E - \chi_{\bigcup_{s\in\Sigma'} Z_s}\right\|_1 \\ &\le \|\chi_E - \chi_K\|_1 + \left\|\chi_K - \chi_{\bigcup_{s\in\Sigma} Z_s}\right\|_1 \\ &< \epsilon + 2\epsilon = 3\epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, we conclude that χ_E lies in the L^1-closure of the linear span of the set

$$\{\chi_A : A \in \mathcal{A}_{s,\sigma,r}(S),\ s \in S_D\}.$$

This proves (ii) and hence the lemma. $\square$

Let X and Y be normed spaces. For $x^* \in X^*$ and $y \in Y$, let $y \otimes x^*$ denote the bounded operator

$$y \otimes x^* \colon X \to Y, \quad x \mapsto x^*(x)y.$$

It is easy to see that $\|y \otimes x^*\| = \|x^*\|\|y\|$.

Let $\mathfrak{X}$ and $\mathfrak{A}$ be normed algebras. If p is an idempotent in $\mathfrak{A}$ and if ϕ is a character on $\mathfrak{X}$, then $p \otimes \phi$ is a continuous homomorphism from $\mathfrak{X}$ to $\mathfrak{A}$. In particular, if $\mathfrak{A}$ is unital, then, for each $\phi \in \Delta(\mathfrak{X})$, $e_{\mathfrak{A}} \otimes \phi$ is a continuous homomorphism from $\mathfrak{X}$ to $\mathfrak{A}$.

We are now in a position to state the main result of this section.

Theorem 6.4. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G, and let $\mathfrak{A}$ be a unital normed algebra. If $\sigma \in \widehat{S}_{\mathrm{u}}$ and if $H \colon L^1(S) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$, then $H = e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$.*

Before presenting the proof, we explain why homomorphisms of the form $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$, $\sigma \in \widehat{S}_{\mathrm{u}}$, arise naturally in Theorem 6.4. Suppose, in the setting of Theorem 6.4, that $H_0 \colon L^1(S) \to \mathfrak{A}$ is a non-zero continuous homomorphism with the following property:

(∗) for every continuous homomorphism $H \colon L^1(S) \to \mathfrak{A}$ satisfying $\|H - H_0\| < 1$, we have $H = H_0$.

Then, by [5, Theorem 2.17], H_0 must necessarily be of the form $H_0 = e_{\mathfrak{A}} \otimes \phi$, where $\phi \in \Delta(L^1(S))$. By Theorem 5.5, we have $\phi = \mathscr{L}_\zeta$ for some $\zeta \in \widehat{S}^\star$, so that $H_0 = e_{\mathfrak{A}} \otimes \mathscr{L}_\zeta$. However, results such as [5, Theorems 4.5, 4.13, 4.19] show that, if $\zeta \in \widehat{S}^\star \setminus \widehat{S}_{\mathrm{u}}$, that is, if ζ is a semi-character, but not a character on S, then $e_{\mathfrak{A}} \otimes \mathscr{L}_\zeta$ may fail to satisfy property (∗). Thus, in Theorem 6.4, it is natural to restrict attention to homomorphisms of the form $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$, $\sigma \in \widehat{S}_{\mathrm{u}}$, where the notation σ emphasises that we are dealing with characters rather than general semi-characters.

Proof of Theorem 6.4. The proof proceeds in several steps.

Step 1. Since $\|\mathscr{L}_\sigma\| = \operatorname{ess\,sup}_{s \in S} |\sigma(s)| = 1$, we have $\|e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| = \|e_{\mathfrak{A}}\|\|\mathscr{L}_\sigma\| = 1$, and since $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$ by assumption, it follows that H is non-zero. Thus, Proposition 4.1 applies, yielding a bounded, strongly continuous semigroup $\{\mathscr{S}(s)\}_{s \in S_D}$ on $\operatorname{Ran}(H)$, determined by H, that satisfies all properties stated in the proposition. If $s \in S_D$, if $\{f_\iota\}_{\iota \in I}$ is as defined in Definition 2.1, if $x \in \operatorname{Ran}(H)$, and if $\iota \in I$, then

$$\begin{aligned} \|H(f_\iota)x - (e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma)(f_\iota)x\| &\le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| \, \|f_\iota\|_1 \|x\| \\ &\le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| \, \|x\|. \end{aligned} \tag{6.1}$$

Since σ is continuous, we have $\lim_{\iota \in I} \mathscr{L}_\sigma(f_\iota) = \sigma(s)$; therefore,

$$\lim_{\iota \in I} (e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma)(f_\iota)x = \sigma(s)x.$$

Combining this with (4.2), we deduce from (6.1) that

$$\|\mathscr{S}(s)x - \sigma(s)x\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| \, \|x\|.$$

Since $|\sigma(s)| = 1$, we further conclude that

$$\|\overline{\sigma(s)}\mathscr{S}(s)x - x\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| \, \|x\|.$$

Therefore

$$\sup_{s \in S_D} \|\overline{\sigma(s)}\mathscr{S}(s) - I_{\operatorname{Ran}(H)}\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|.$$

This, combined with the assumption that $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$, yields

$$\sup_{s \in S_D} \|\overline{\sigma(s)}\mathscr{S}(s) - I_{\operatorname{Ran}(H)}\| < 1.$$

An application of Proposition 6.2 now implies that $\overline{\sigma(s)}\mathscr{S}(s) = I_{\mathrm{Ran}(H)}$ for all $s \in S_D$. Hence, immediately, $\mathscr{S}(s) = \sigma(s) I_{\mathrm{Ran}(H)}$ for all $s \in S_D$, and furthermore, in view of (4.3),

$$H(f)x = \mathscr{L}_\sigma(f)x \tag{6.2}$$

for all $f \in L^1(S)$ and all $x \in \mathrm{Ran}(H)$.

STEP 2. Let $f, g \in L^1(S)$ be such that $f \star g = g \star f$ and $\mathscr{L}_\sigma(f) \neq 0$ and $\mathscr{L}_\sigma(g) \neq 0$. We claim that

$$(\mathscr{L}_\sigma(f))^{-1} H(f) = (\mathscr{L}_\sigma(g))^{-1} H(g). \tag{6.3}$$

Applying (6.2) with x replaced by $H(g)$, we obtain

$$H(f)H(g) = \mathscr{L}_\sigma(f)H(g). \tag{6.4}$$

Reversing the roles of f and g in the above equality, we also obtain

$$H(g)H(f) = \mathscr{L}_\sigma(g)H(f).$$

Since

$$H(f)H(g) = H(f \star g) = H(g \star f) = H(g)H(f),$$

we conclude that

$$\mathscr{L}_\sigma(f)H(g) = \mathscr{L}_\sigma(g)H(f).$$

This immediately yields (6.3), as claimed.

Let $g \in L^1(S)$ be such that $\mathscr{L}_\sigma(g) \neq 0$. Define

$$p = (\mathscr{L}_\sigma(g))^{-1} H(g).$$

We claim that p is an idempotent. Indeed, g and $g \star g$ commute as elements of $L^1(S)$ (because convolution is associative). Moreover, $\mathscr{L}_\sigma(g \star g) = (\mathscr{L}_\sigma(g))^2 \neq 0$. We may therefore apply (6.3) with $f = g \star g$ to find that

$$(\mathscr{L}_\sigma(g \star g))^{-1} H(g \star g) = (\mathscr{L}_\sigma(g))^{-1} H(g).$$

But

$$(\mathscr{L}_\sigma(g \star g))^{-1} H(g \star g) = ((\mathscr{L}_\sigma(g))^2)^{-1} (H(g))^2 = \big((\mathscr{L}_\sigma(g))^{-1} H(g)\big)^2,$$

and this means that p is an idempotent. The claim follows.

STEP 3. Let $\alpha := \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|$. By assumption, $\alpha < 1$. Choose β so that $\alpha < \beta < 1$. For any $A \in \mathcal{M}^{\mathrm{fin}}(S)$ with $m_G(A) > 0$, set

$$f_A = (m_G(A))^{-1} \chi_A.$$

Since $0 < m_G(A) < \infty$, the function f_A is well defined. Let $s \in S_D$. By Lemma 6.3, $\mathcal{A}_{s,\sigma,1-\beta}(S)$ is non-empty. Select $A \in \mathcal{A}_{s,\sigma,1-\beta}(S)$. By the definition of $\mathcal{A}_{s,\sigma,1-\beta}(S)$, $0 < m_G(A) < \infty$, so, in particular, f_A is well defined. Since

$$\begin{aligned} |\mathscr{L}_\sigma(f_A) - \sigma(s)| &= \left| \frac{1}{m_G(A)} \int_A (\sigma(t) - \sigma(s)) \,\mathrm{d}m_G(t) \right| \\ &\leq \frac{1}{m_G(A)} \int_A |\sigma(t) - \sigma(s)| \,\mathrm{d}m_G(t) \leq \sup_{t \in A} |\sigma(t) - \sigma(s)|, \end{aligned}$$

and since, by the definition of $\mathcal{A}_{s,\sigma,1-\beta}(S)$, $\sup_{t \in A} |\sigma(t) - \sigma(s)| \leq 1 - \beta$, we have

$$|\mathscr{L}_\sigma(f_A)) - \sigma(s)| \leq 1 - \beta. \tag{6.5}$$

Note that $\mathscr{L}_\sigma(f_A) \neq 0$. Indeed, otherwise, (6.5) would imply $|\sigma(s)| \leq 1 - \beta$, contradicting $|\sigma(s)| = 1$.

Set

$$p_A = (\mathscr{L}_\sigma(f_A))^{-1} H(f_A).$$

By Step 2, p_A is an idempotent.

We claim that $\|p_A - e_{\mathfrak{A}}\| < 1$. Indeed, since

$$|1 - \mathscr{L}_\sigma(f_A)\overline{\sigma(s)}| = |\mathscr{L}_\sigma(f_A) - \sigma(s)| \le 1 - \beta < 1,$$

the series

$$\sum_{k=0}^{\infty}(1 - \mathscr{L}_\sigma(f_A)\overline{\sigma(s)})^k$$

converges absolutely, with sum $\big(\mathscr{L}_\sigma(f_A)\overline{\sigma(s)}\big)^{-1}$. Furthermore,

$$\frac{1}{|\mathscr{L}_\sigma(f_A)|} = \frac{1}{|\mathscr{L}_\sigma(f_A)\overline{\sigma(s)}|} \le \sum_{k=0}^{\infty}|1 - \mathscr{L}_\sigma(f_A)\overline{\sigma(s)}|^k \le \sum_{k=0}^{\infty}(1-\beta)^k = \beta^{-1}.$$

Hence

$$\begin{aligned}\|p_A - e_{\mathfrak{A}}\| &= \frac{1}{|\mathscr{L}_\sigma(f_A)|}\|H(f_A) - \mathscr{L}_\sigma(f_A)e_{\mathfrak{A}}\| \le \frac{1}{|\mathscr{L}_\sigma(f_A)|}\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|\,\|f_A\|_1 \\ &\le \beta^{-1}\alpha < 1.\end{aligned}$$

The claim is established.

By the definition of p_A,

$$H(f_A) = \mathscr{L}_\sigma(f_A)p_A$$

Since p_A is an idempotent and $\|p_A - e_{\mathfrak{A}}\| < 1$, Proposition 6.1 implies that $p_A = e_{\mathfrak{A}}$. Hence

$$H(f_A) = \mathscr{L}_\sigma(f_A)e_{\mathfrak{A}}.$$

By the linearity and continuity of both H and $\mathscr{L}_\sigma$, this equality extends to

$$H(f) = \mathscr{L}_\sigma(f)e_{\mathfrak{A}}, \tag{6.6}$$

where f belongs to the L^1-closure of the linear span of

$$\{f_A : A \in \mathcal{A}_{s,\sigma,1-\beta}(S),\ s \in S_D\}.$$

It is clear that this closed linear span coincides with the L^1-closure of the linear span of

$$\{\chi_A : A \in \mathcal{A}_{s,\sigma,1-\beta}(S),\ s \in S_D\}.$$

By Lemma 6.3, the latter space equals $L^1(S)$. Therefore, (6.6) holds for all f in $L^1(S)$, which means that $H = e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$, completing the proof. $\square$

Theorem 6.4 generalises [5, Theorem 3.5] by removing the assumption that e_G, the neutral element of G, lies in $D(S)$, the density point set of S. This condition in [5] ensures that $L^1(S)$ admits an approximate identity of bound 1, a crucial requirement for the proof given therein.

To highlight the significance and illustrate the scope of Theorem 6.4, we now present two examples of semigroup algebras that do not admit an approximate identity—let alone one of bound 1.

Example 6.5. Consider $\mathbb{R}$ as a group under addition, equipped with its usual topology. Let $a > 0$. Then $[a, \infty)$ is a Lebesgue measurable subsemigroup of $\mathbb{R}$. The semigroup algebra of $[a, \infty)$, denoted for brevity by $L^1(a, \infty)$—rather than $L^1([a, \infty))$, as one might expect—can be described explicitly as the Banach algebra of equivalence classes of complex-valued, Lebesgue integrable functions on $[a, \infty)$ that are equal almost everywhere, equipped with the norm

$$\|f\|_1 = \int_a^\infty |f(s)|\,\mathrm{d}s \quad (f \in L^1(a, \infty))$$

and the convolution product

$$(f \star g)(s) = \begin{cases}\int_a^{s-a} f(s-t)g(t)\,\mathrm{d}t & \text{for a. e. } s \ge 2a,\\ 0 & \text{for a. e. } a \le s < 2a\end{cases} \quad (f, g \in L^1(a, \infty)).$$

We now show that the algebra $L^1(a,\infty)$ lacks an approximate identity. Let $E \subset [a, 2a)$ be a Lebesgue measurable set of positive measure. Define a linear functional ϕ on $L^1(a,\infty)$ by

$$\phi(f) = \int_E f(s)\,\mathrm{d}s \quad (f \in L^1(a,\infty));$$

it has norm 1. For all $e, f \in L^1(a,\infty)$, $e \star f$ vanishes almost everywhere on $[a, 2a)$, and hence $\phi(e \star f) = 0$. Therefore,

$$|\phi(f)| = |\phi(f) - \phi(e \star f)| \leq \|f - e \star f\|_1$$

for all $e, f \in L^1(a,\infty)$. Thus, if $f \in L^1(a,\infty)$ is such that $\phi(f) \neq 0$, then there exists no $e \in L^1(a,\infty)$ such that $\|f - e \star f\|_1 < |\phi(f)|$. This precludes the existence of an approximate identity in $L^1(a,\infty)$.

Example 6.6. Consider $\mathbb{Z}$ as a group under addition, equipped with the discrete topology. Let $k \in \mathbb{N}$, and set $S = \{k, k+1, k+2, \dots\}$. Then S is a subsemigroup of $\mathbb{Z}$. The semigroup algebra $\ell^1(S)$ is the Banach algebra of all complex-valued, absolutely summable sequences $\{x_n\}_{n \geq k}$, with the norm

$$\|x\|_1 = \sum_{n=k}^{\infty} |x_n| \quad (x = \{x_n\}_{n \geq k} \in \ell^1(S))$$

and the convolution product

$$(x \star y)_m = \begin{cases} \sum_{n=k}^{m-k} x_{m-n} y_n & \text{for } m \geq 2k, \\ 0 & \text{for } k \leq m < 2k \end{cases}$$
$$(x = \{x_n\}_{n \geq k} \in \ell^1(S), \quad y = \{y_n\}_{n \geq k} \in \ell^1(S)).$$

The algebra $\ell^1(S)$ lacks an approximate identity. This follows by adapting the argument from Example 6.5: select a non-empty subset $E \subset \{k, \dots, 2k-1\}$, define the norm-one functional ϕ on $\ell^1(S)$ by

$$\phi(x) = \sum_{m \in E} x_m \quad (x = \{x_n\}_{n \geq k} \in \ell^1(S)),$$

and observe that $\phi(e \star x) = 0$ for all $e, x \in \ell^1(S)$, leading to the same contradiction as in Example 6.5, and hence precluding the existence of an approximate identity in $\ell^1(S)$.

We conclude by recalling two concepts from [5] and presenting a consequence of Theorem 6.4.

Let $\mathfrak{X}$ and $\mathfrak{A}$ be normed algebras. We denote by $\operatorname{Hom}(\mathfrak{X}, \mathfrak{A})$ the set of all continuous algebra homomorphisms from $\mathfrak{X}$ to $\mathfrak{A}$.

Definition 6.7. Let $\mathfrak{X}$ be a normed algebra, let $\mathfrak{A}$ be a unital normed algebra, and let $H \in \operatorname{Hom}(\mathfrak{X}, \mathfrak{A})$. Then:

(i) H is *isolated* if there does not exist a sequence $\{H_n\}_{n \in \mathbb{N}}$ in $\operatorname{Hom}(\mathfrak{X}, \mathfrak{A})$ such that $H_n \neq H$ for all $n \in \mathbb{N}$, and $\|H - H_n\| \to 0$ as $n \to \infty$;

(ii) H is *totally isolated* if, for every unital normed algebra $\mathfrak{B}$ and every isometric homomorphism $I \colon \mathfrak{A} \to \mathfrak{B}$ with $I(e_{\mathfrak{A}}) = e_{\mathfrak{B}}$, there does not exist a sequence $\{H_n\}_{n \in \mathbb{N}}$ in $\operatorname{Hom}(\mathfrak{X}, \mathfrak{B})$ such that $H_n \neq I \circ H$ for all $n \in \mathbb{N}$, and $\|I \circ H - H_n\| \to 0$ as $n \to \infty$.

With these preparations in place, we can now state the following theorem, which follows directly from Theorem 6.4.

Theorem 6.8. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G, and let $\mathfrak{A}$ be a unital normed algebra. If $\sigma \in \widehat{S}_{\mathrm{u}}$, then $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma \colon L^1(S) \to \mathfrak{A}$ is totally isolated.*

This theorem generalises [5, Theorem 3.7], which assumes that $e_G \in D(S)$. The proof of Theorem 6.8 is essentially identical to that of [5, Theorem 3.7], and proceeds as follows.

Proof of Theorem 6.8. Let $\mathfrak{B}$ be a unital normed algebra, and let $I \colon \mathfrak{A} \to \mathfrak{B}$ be an isometric homomorphism with $I(e_{\mathfrak{A}}) = e_{\mathfrak{B}}$. Then $I \circ (e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma) = e_{\mathfrak{B}} \otimes \mathscr{L}_\sigma$. By Theorem 6.4, if $H \in \operatorname{Hom}(L^1(S), \mathfrak{B})$ is such that $\|H - e_{\mathfrak{B}} \otimes \mathscr{L}_\sigma\| < 1$, then $H = e_{\mathfrak{B}} \otimes \mathscr{L}_\sigma$. From this observation, the theorem follows immediately. □

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