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Some Non-trivial Cocycles

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For each locally compact non-compact σ-compact Abelian group G whose dual is an I-group and each compact Abelian group Σ such that there is a one-to-one continous homomorphism from G onto a dense subgroup of Σ, we show the existence of cocycles on Σ whose associated unitary representations of G have purely singular continuous spectrum, as well as the existence of cocycles on Σ whose associated unitary representations of G have purely Haar spectrum.

INTRODUCTION

Let G be a locally compact non-compact Abelian group and Σ be a compact Abelian group. Suppose that there is a one-to-one continuous homomorphism α from G onto a dense subgroup of Σ.

Call a function with values of unit modulus a unitary function.

A cocycle on Σ is a Borel unitary function A on $\Sigma \times G$ satisfying, for all $\sigma \in \Sigma$ and all $g, g' \in G$, the cocycle identity

$$A(\sigma, g+g') = A(\sigma, g)\, A(\sigma + \alpha(g), g').$$

A cocycle A is trivial if there exist an element γ in the dual group $\hat{G}$ of G and a Borel unitary function B on Σ such that, for each $g \in G$, the identity

$$A(\sigma, g) = (g, \gamma)\, B(\sigma)\, \overline{B(\sigma + \alpha(g))}$$

holds for m_Σ-almost all σ in Σ, m_Σ being the Haar measure in Σ.

For $1 \leqslant p \leqslant +\infty$, let $L^p(\Sigma)$ denote the pth Lebesgue space based on m_Σ.

Given $g \in G$, by setting

$$(U(g)\, \varphi)(\sigma) = A(\sigma, g)\, \varphi(\sigma + \alpha(g)) \qquad (\varphi \in L^2(\Sigma), \sigma \in \Sigma)$$

one defines a unitary operator $U(g)$ in $L^2(\Sigma)$. The map $U: g \to U(g)$ is a

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strongly continuous unitary representation of G in $L^2(\Sigma)$. By the Stone–Naĭmark–Ambrose–Godement theorem (cf. [2, Theorem 6.2.1]), there is a unique projection-valued measure P on the Borel σ-algebra of $\hat{G}$, taking values in a Boolean algebra of projections in $L^2(\Sigma)$, such that for each $g \in G$,

$$U(g) = \int_{\hat{G}} (g, -\gamma)\, dP_\gamma,$$

where the integral is to be interpreted in the sense of strong convergence. It turns out that P is spectrally homogeneous: its spectrum is either pure point, or purely singular continuous, or purely Haar. Moreover, P has pure point spectrum if and only if A is trivial.

One can naturally ask whether there exist, for a given pair of groups (G, Σ), cocycles on Σ whose associated unitary representations of G have continuous spectrum. The answer to this question is of interest not only in itself, but actually various problems in analysis devolve to the consideration of non-trivial cocycles. In ergodic theory, cocycles giving continuous spectrum arise in connection with group extensions and velocity changes (cf. [21, 22]), in harmonic analysis such cocycles arise in connection with invariant subspaces on compact solenoidal groups (cf. [7, 9]), and in differential equations they appear in the context of the quasi-momentum analysis of pseudodifferential operators with spatially almost periodic symbols (cf. [4, 5, 14]). Up to now the existence of non-trivial cocycles has been established only for a limited number of classes of pairs (G, Σ). For the pairs $(\mathbb{R}, \Sigma)$, H. Helson and J.-P. Kahane [11] constructed cocycles giving singular continuous spectrum, while for the pairs $(\mathbb{Z}, \Sigma)$, Helson and W. Parry [13] established the existence of cocycles giving Lebesgue spectrum. A refinement of the Gamelin isomorphism theorem (cf. [1, 6]) applied to these results yields, for the pairs $(\mathbb{R}, \Sigma)$, the existence of cocycles giving Lebesgue spectrum, and, for the pairs $(\mathbb{Z}, \Sigma)$, the existence of cocycles giving singular continous spectrum. For the pairs (G, Σ) in which G is a discrete countable group, J. Mathew and M. G. Nadkarni [17] established the existence of cocycles giving Haar spectrum. Some other results pertaining to the construction of non-trivial cocycles can be found in [4, 6–10, 12, 16, 18, 20, 22, 26].

In this paper we establish the existence of cocycles giving singular continuous and Haar spectra for the pairs (G, Σ) in which G is a locally compact non-compact σ-compact Abelian group whose dual is an I-group. Thereby, we generalize the results of Helson and Kahane and of Helson and Parry, and the consequences of these via Gamelin's theorem, and provide an alternative approach to a wide class of cases covered by the result of Mathew and Nadkarni. Our results will have a somewhat nonconstructive character—we shall find cocycles giving two kinds of contInous spectrum among sample elements of certain random cocycles. However, any of such sample cocycles will have a particularly regular form—it will be a continuous function on $\Sigma \times G$ explicitly expressible in terms of harmonic-analytic objects associated with G and Σ.

The paper is organized as follows. Following a preliminary section in which we set the context for the subsequent development, we begin in Section 2 by proving that random cocycles give a spectrum of non-random type. In Section 3 we exhibit a class of non-trivial random cocycles. In Section 4 we show that this class contains a subclass consisting of random cocycles giving singular continuous spectrum. We bring the paper to an end with a fifth section in which we reveal a subclass of the class from Section 3 made up of random cocycles giving Haar spectrum.

1. Preliminaries

Throughout the paper G will be a locally compact non-compact Abelian group, Σ a compact Abelian group, and α a one-to-one continuous homomorphism from G onto a dense subgroup of Σ. In the subsequent sections some additional conditions on G will be imposed.

For each σ in Σ, denote by T_σ the translation operator by σ acting on $L^\infty(\Sigma)$, and denote by T the representation $g \to T_{\alpha(g)}$ of G in the automorphism group of $L^\infty(\Sigma)$.

Let ρ be the multiplication representation of $L^\infty(\Sigma)$ on $L^2(\Sigma)$ given by

$$\rho(f)\,\varphi = f\varphi \qquad (f \in L^\infty(\Sigma),\ \varphi \in L^2(\Sigma)).$$

Let A be a cocycle on Σ and U be the associated unitary representation of G. The pair (ρ, U) constitutes a so-called covariant representation of the W^*-dynamical system $(L^\infty(\Sigma), G, T)$—for each $f \in L^\infty(\Sigma)$ and each $g \in G$, one has

$$U(g)\,\rho(f)\,U(g)^* = \rho(T_{\alpha(g)} f), \tag{1.1}$$

where $U(g)^*$ denotes the adjoint to $U(g)$. This covariant representation is irreducible: any bounded linear operator R in $L^2(\Sigma)$, commuting with each $U(g)$ $(g \in G)$ and each $\rho(f)$ $(f \in L^\infty(\Sigma))$, is scalar. In fact, the commutant of $\rho(L^\infty(\Sigma))$ being $\rho(L^\infty(\Sigma))$ itself (cf. [23, Theorem 3.1.2]), we have $R = \rho(f)$ for some $f \in L^\infty(\Sigma)$. By (1.1), $f = T_{\alpha(g)} f$ for each $g \in G$. Since the action $(g, \sigma) \to \sigma + \alpha(g)$ of G on Σ is ergodic, f is a constant function and, accordingly, R is a scalar operator.

Let $M(\hat{G})$ be the space of all bounded complex-valued regular Borel measures on $\hat{G}$. $M(\hat{G})$ is a complex Banach lattice and, under convolution denoted as $*$, a Banach algebra.

A band in $M(\hat{G})$ (in the vector-lattice-theoretic sense) is a subset I of $M(\hat{G})$ such that

(i) if $\mu \in I$ and a measure $\nu \in M(\hat{G})$ is absolutely continuous with respect to $|\mu|$, then $\nu \in I$;

(ii) if $(\mu_n)_{n \in \mathbb{N}}$ is a sequence in I with $\sum_{n=1}^{\infty} \|\mu_n\| < +\infty$, then $\sum_{n=1}^{\infty} \mu_n \in I$.

Let $\hat{\alpha}$ be the homomorphism from $\hat{\Sigma}$ into $\hat{G}$ given by

$$(g, \hat{\alpha}(\chi)) = (\alpha(g), \chi) \qquad (g \in G, \chi \in \hat{\Sigma}).$$

A band I in $M(\hat{G})$ is $\hat{\Sigma}$-invariant if for each $\chi \in \hat{\Sigma}$, I contains along with any measure μ the measure $\mu * \delta_{\hat{\alpha}(\chi)}$; here $\delta_{\hat{\alpha}(\chi)}$ stands for the Dirac measure concentrated at $\hat{\alpha}(\chi)$. A $\hat{\Sigma}$-invariant band I in $M(\hat{G})$ is ergodic if for each $\hat{\Sigma}$-invariant subband J of I, either $J = \{0\}$ or $J = I$.

Let P be the projection-valued measure associated with the representation U. P constitutes a so-called system of imprimitivity for ρ based on $\hat{G}$—Eq. (1.1) implies that for each $\chi \in \hat{\Sigma}$ and each Borel subset E of $\hat{G}$,

$$\rho(\chi)^* P_E \rho(\chi) = P_{E + \hat{\alpha}(\chi)}, \tag{1.2}$$

where $E + \hat{\alpha}(\chi) = \{\gamma \in \hat{G} : \gamma = \eta + \hat{\alpha}(\chi), \eta \in E\}$. For each $\varphi, \psi \in L^2(\Sigma)$, let $\pi_{\varphi,\psi}$ be the measure on $\hat{G}$ defined by

$$\pi_{\varphi,\psi}(E) = (P_E \varphi, \psi) \qquad (E \text{ a Borel subset of } \hat{G}),$$

where $(\cdot, \cdot)$ denotes the scalar product of $L^2(\Sigma)$. It is well known that $I = \{\pi_{\varphi,\psi} : \varphi, \psi \in L^2(\Sigma)\}$ is a band in $M(\hat{G})$ (cf. [25]). By (1.2), $\pi_{\varphi,\psi} * \delta_{\hat{\alpha}(\chi)} = \pi_{\rho(-\chi)\varphi, \rho(-\chi)\psi}$ for each $\varphi, \psi \in L^2(\Sigma)$ and each $\chi \in \hat{\Sigma}$, so I is $\hat{\Sigma}$-invariant. Actually, I is ergodic. To see this, suppose J is a non-zero $\hat{\Sigma}$-invariant subband of I. The positive cone of J being non-zero and each of its members taking the form $\pi_{\varphi,\varphi}$ for some $\varphi \in L^2(\Sigma)$, the set $H = \{\varphi \in L^2(\Sigma) : \pi_{\varphi,\varphi} \in J\}$ is a non-zero closed subspace of $L^2(\Sigma)$. Let R be the orthogonal projection in $L^2(\Sigma)$ with the range space H. In view of (1.2), R commutes with each $\rho(\chi)$ $(\chi \in \hat{\Sigma})$. Clearly, R commutes also with each $U(g)$ $(g \in G)$. Since the von Neumann algebra generated by the $\rho(\chi)$ coincides with $\rho(L^\infty(\Sigma))$, the irreducibility of the covariant representation (ρ, U) ensures now that R is the identity operator. Hence $I = J$, as was to be shown.

$l^1(\hat{G})$, $L^1(\hat{G})$, and the space $M_{sc}(\hat{G})$ of the continuous measures in $M(\hat{G})$ singular with respect to Haar measure are all $\hat{\Sigma}$-invariant bands in $M(\hat{G})$. As these bands are pairwise disjoint (in the vector-lattice-theoretic sense) and their join is the whole $M(\hat{G})$, one of them must intersect I outside zero. Since I is ergodic, the band intersecting $I\backslash\{0\}$ actually contains I.

When $I \subset l^1(\hat{G})$, P is said to have pure point spectrum. This is the case exactly when A is trivial. For if A can be represented as a trivial cocycle by means of a Borel unitary function B defined on Σ and an element γ of $\hat{G}$, then $U(g)\, B = (g, \gamma)\, B$ for each $g \in G$; consequently, $\pi_{B,B} = \delta_{-\gamma}$ and, by the preceding paragraph, $I \subset l^1(\hat{G})$. Conversely, if P has pure point spectrum, then $\pi_{B,B} = \delta_{-\gamma}$ for some $B \in L^2(\Sigma)$ of unit norm and some $\gamma \in \hat{G}$, hence $U(g)\, B = (g, \gamma)\, B$ for all $g \in G$. By (1.1), for each $f \in L^\infty(\Sigma)$ and each $g \in G$,

$$(\rho(f)\, B, B) = (\rho(f)\, U(g)^*\, B, U(g)^*\, B) = (\rho(T_{\alpha(g)} f)\, B, B),$$

hence $T_{\alpha(g)}(|B|) = |B|$ for each $g \in G$. As the G action on Σ is ergodic, $|B|$ is m_Σ-essentially constant, and so B can be taken to be unitary. The triviality of A follows.

When $I \subset M_{sc}(\hat{G})$, P is said to have purely singular continuous spectrum and A is said to be of singular continuous type.

If $I \subset L^1(\hat{G})$, then in fact $I = L^1(G)$, for $L^1(G)$ is an ergodic band; in this case P is said to have purely Haar spectrum and A is said to be of Haar type.

When either $I \subset M_{sc}(\hat{G})$ or $I \subset L^1(\hat{G})$, P is said to have purely continuous spectrum. This is the case exactly when A is non-trivial.

Let $M_0(\hat{G})$ be the space of all measures in $M(\hat{G})$ whose Fourier transforms vanish at infinity. $M_0(\hat{G})$ is a $\hat{\Sigma}$-invariant band in $M(\hat{G})$, and so is $M_0(\hat{G})^\perp$, the vector-lattice-theoretic disjoint complement of $M_0(\hat{G})$. When $I \subset M_0(\hat{G})$, we say that A is of type (C_0); when $I \subset M_0(\hat{G})^\perp$, we say that A is of oscillatory type. It is clear that any cocycle of type (C_0) is non-trivial, and that any non-trivial cocycle of oscillatory type is of singular continuous type.

2. A Purity Theorem

Throughout this section we shall suppose that G is a locally compact non-compact σ-compact Abelian group.

Given a topological space X, we denote by $\mathscr{B}(X)$ the Borel σ-algebra of X.

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. By a random cocycle $\{A_\omega\}$ on Σ we mean a unitary $(\mathscr{A} \otimes \mathscr{B}(\Sigma) \otimes \mathscr{B}(G), \mathscr{B}(\mathbb{T}))$-measurable function $(\omega, \sigma, g) \to A_\omega(\sigma, g)$ on $\Omega \times \Sigma \times G$ such that for each $\omega \in \Omega$, $A_\omega: (\sigma, g) \to A_\omega(\sigma, g)$ is a cocycle on Σ. A random cocycle $\{A_\omega\}$ is said to be trivial (non-trivial, of singular continuous type, etc.) if for $\mathbb{P}$-almost all ω in Ω, A_ω is a trivial (non-trivial, of singular continuous type, etc.) cocycle on Σ.

THEOREM 2.1. *Let $(\mathscr{A}_n)_{n\in\mathbb{N}}$ be a sequence of independent σ-algebras of $\mathscr{A}$, $\omega\to\gamma_\omega$ be a $(\mathscr{A}, \mathscr{B}(\hat{G}))$-measurable function from Ω into $\hat{G}$, and, for each $n\in\mathbb{N}$, $(\omega,\sigma)\to B_\omega^{(n)}(\sigma)$ be a unitary $(\mathscr{A}_n\otimes\mathscr{B}(\Sigma), \mathscr{B}(\mathbb{T}))$-measurable function on $\Omega\times\Sigma$. Suppose that for each (ω,σ,g) in $\Omega\times\Sigma\times G$, the product*

$$A_\omega(\sigma,g)=\prod_{k=1}^{\infty}(g,\gamma_\omega)\,B_\omega^{(k)}(\sigma)\,\overline{B_\omega^{(k)}(\sigma+\alpha(g))}$$

converges. Then $\{A_\omega\}$: $(\omega,\sigma,g)\to A_\omega(\sigma,g)$ is a random cocycle on Σ that is either trivial or non-trivial, either of type (C_0) or of oscillatory type.

Proof. That the map $\{A_\omega\}$ is a random cocycle is clear.

We begin the proof of the first alternative by recalling that on the space $W(G)$ of all weakly almost periodic functions on G (containing in particular the space of all the Fourier transforms of measures in $M(\hat{G})$), there is defined a unique positive normalized translation-invariant functional M (cf. [3, Corollary 1.26]). Moreover, the σ-compactness of G ensures the existence of a sequence $(H_n)_{n\in\mathbb{N}}$ of compact subsets of G such that for each f in $W(G)$,

$$M(f)=\lim_{n\to\infty}\frac{1}{m_G(H_n)}\int_{H_n}f(g)\,dm_G(g) \tag{2.1}$$

(cf. [15, Theorem 18.14; 3, Note 3.20]); in the sequel, we shall denote $M(f)$ also by $M_g(f(g))$.

For each $\omega\in\Omega$, let U_ω denote the unitary representation of G associated with A_ω. Given $n\in\mathbb{N}$ and $\omega\in\Omega$, let $U_\omega^{(n)}$ be the unitary representation of G associated with the cocycle

$$(\sigma,g)\to\prod_{k=n+1}^{\infty}(g,\gamma_\omega)\,B_\omega^{(k)}(\sigma)\,\overline{B_\omega^{(k)}(\sigma+\alpha(g))}$$

and, for each $\sigma\in\Sigma$, put

$$Z_\omega^{(n)}(\sigma)=\prod_{k=1}^{n}B_\omega^{(k)}(\sigma).$$

For each $\omega\in\Omega$, the purity of the spectrum of the projection-valued measure associated with U_ω ensures that the cocycle A_ω is non-trivial if and only if $g\to(U_\omega(g)\,1,1)$ is the Fourier transform of a continuous measure on $\hat{G}$. By Wiener's theorem (cf. [24, Theorem 5.6.9]), the latter condition is equivalent to

$$M_g(|(U_\omega(g)\,1,1)|^2)=0. \tag{2.2}$$

Clearly, for each $n \in \mathbb{N}$, we have

$$|(U_\omega(g)\,1, 1)| = |(U_\omega^{(n)}(g)\,Z_\omega^{(n)}, Z_\omega^{(n)})|,$$

so (2.2) holds if and only if

$$M_g(|(U_\omega^{(n)}(g)\,Z_\omega^{(n)}, Z_\omega^{(n)})|^2) = 0.$$

By the purity of the spectrum of the projection-valued measure associated with $U_\omega^{(n)}$, the last relation is in turn equivalent to

$$M_g(|(U_\omega^{(n)}(g)\,1, 1)|^2) = 0. \tag{2.3}$$

In view of (2.1), the set

$$\Omega' = \{\omega \in \Omega\colon M_g(|(U_\omega(g)\,1, 1)|^2) = 0\}$$

is $\mathscr{A}$-measurable and, by the equivalence of (2.2) and (2.3), for each $n \in \mathbb{N}$, it in fact belongs to the σ-algebra generated by all $\mathscr{A}_k$ with $k \geqslant n$. By Kolmogorov's zero–one law, we have either $\mathbb{P}(\Omega') = 0$ or $\mathbb{P}(\Omega') = 1$. Accordingly, $\{A_\omega\}$ is either trivial or non-trivial.

The proof of the second alternative is similar and is based on the observation that, given any $\omega \in \Omega$, the function $g \to (U_\omega(g)\,1, 1)$ vanishes at infinity if and only if, for each $n \in \mathbb{N}$, the function $g \to (U_\omega^{(n)}(g)\,1, 1)$ does. The details are left to the reader.

3. Non-trivial Random Cocycles

Unless stated otherwise, from now onwards G will be a σ-compact non-compact Abelian group whose dual is an I-group. We recall that a locally compact Abelian group is an I-group if every neighbourhood of 0 in this group contains an element of infinite order (cf. [24, p. 46]).

In this section we shall exhibit a class of non-trivial random cocycles on Σ.

Given a Lebesgue integrable function h on the unit circle $\mathbb{T}$ and an integer n, we denote by $\hat{h}(n)$ the nth Fourier coefficient of h, i.e.,

$$\hat{h}(n) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{iu})\, e^{-inu}\, du.$$

Given a continuous function h on $\mathbb{T}$, we denote by ω_h the modulus of continuity of h defined, for any $\delta > 0$, by

$$\omega_h(\delta) = \sup\{|h(e^{i(t+\Theta)}) - h(e^{it})|\colon 0 \leqslant t < 2\pi,\ 0 \leqslant \Theta < 2\pi,\ |e^{i\Theta} - 1| < \delta\}.$$

By a standard unitary function on $\mathbb{T}$ we mean a unitary continuous function on $\mathbb{T}$ with at least two non-zero Fourier coefficients.

A sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G will be called standard if

$$K_1 \subset K_2 \subset \cdots \subset \bigcup_{n=1}^{\infty} K_n = G$$

and, for each $n \in \mathbb{N}$, the interior of K_{n+1} contains K_n.

Given an element a of a group, we write $\operatorname{ord} a$ for the order of a.

For a function h defined on $\mathbb{T}$ and χ in $\hat{\Sigma}$, we denote by f_χ the superposition $f \circ \chi$.

Let f be a standard unitary function on $\mathbb{T}$, $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \omega_f(\delta_n) < +\infty$, $(K_n)_{n\in\mathbb{N}}$ be a standard sequence of compact subsets of G, and i be a function from $\mathbb{N}$ into itself such that $\lim_{n\to\infty} i(n) = +\infty$. Assume $(\chi_n)_{n\in\mathbb{N}}$ is a sequence in $\hat{\Sigma}$ such that

(i) $\lim_{n\to\infty} \operatorname{ord} \chi_n = +\infty$,

(ii) for each $n \in \mathbb{N}$, if $g \in K_{i(n)}$, then

$$|(\alpha(g), \chi_n) - 1| < \delta_n.$$

The existence of $(\chi_n)_{n\in\mathbb{N}}$, given $(\delta_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$, and i, is seen as follows. Because $\hat{G}$ is an I-group, for each $n \in \mathbb{N}$, there is γ_n in $\hat{G}$ of infinite order with

$$|(g, \gamma_n) - 1| < \delta_n$$

for all g in $K_{i(n)}$. The group $\alpha(G)$ is dense in Σ and the homomorphism α is one-to-one, so the group $\hat{\alpha}(\hat{\Sigma})$ is dense in $\hat{G}$. Accordingly, for each $n \in \mathbb{N}$, there exists $\chi_n \in \hat{\Sigma}$ of order $> n$ such that

$$|(\alpha(g), \chi_n) - 1| < \delta_n$$

for all g in $K_{i(n)}$.

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space carrying a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of independent random variables, each uniformly distributed on Σ. One provides an example of such a probability space with a corresponding sequence of random variables by taking the direct product $\Sigma^{\mathbb{N}}$ for Ω, the Borel σ-algebra of $\Sigma^{\mathbb{N}}$ for $\mathscr{A}$, the Haar measure in $\Sigma^{\mathbb{N}}$ for $\mathbb{P}$, and, for each $n \in \mathbb{N}$, the projection from $\Sigma^{\mathbb{N}}$ onto the nth copy of Σ for σ_n.

For each $(\omega, \sigma, g) \in \Omega \times \Sigma \times G$, put

$$A_\omega(\sigma, g) = \prod_{n=1}^{\infty} f_{\chi_n}(\sigma + \sigma_n(\omega))\overline{f_{\chi_n}(\sigma + \sigma_n(\omega) + \alpha(g))}.$$

The product is convergent with uniform convergence in ω and σ, and locally uniform convergence in g. In fact, given any compact subset K of G, there exists an integer n_0 such that $K \subset K_{i(n)}$ for all $n \geqslant n_0$; hence, for each $(\omega, \sigma, g) \in \Omega \times \Sigma \times K$ and each $n \geqslant n_0$,

$$\begin{aligned} &|f_{\chi_n}(\sigma + \sigma_n(\omega)) \overline{f_{\chi_n}(\sigma + \sigma_n(\omega) + \alpha(g))} - 1| \\ &\quad = |f_{\chi_n}(\sigma + \sigma_n(\omega) + \alpha(g)) - f_{\chi_n}(\sigma + \sigma_n(\omega))| \\ &\quad \leqslant \omega_f(\delta_n). \end{aligned}$$

It is clear that, for any $\omega \in \Omega$, $A_\omega: (\sigma, g) \to A_\omega(\sigma, g)$ is a cocycle on Σ, which is a continuous function on $\Sigma \times G$; accordingly, $\{A_\omega\}$: $(\omega, \sigma, g) \to A_\omega(\sigma, g)$ is a random cocycle on Σ.

The main result of this section is the following.

THEOREM 3.1. *The random cocycle* $\{A_\omega\}$ *is non-trivial.*

Before giving the proof, we require one preliminary.

For any continuous function h on $\mathbb{T}$ and any $0 \leqslant u < 2\pi$, we put

$$\psi_h(e^{iu}) = \frac{1}{4\pi^2} \int_0^{2\pi} \left| \int_0^{2\pi} h(e^{i(s+t)}) \overline{h(e^{it})} \, \overline{h(e^{i(s+t+u)})} \, h(e^{i(t+u)}) \, dt \right|^2 ds;$$

we let, moreover, $S_h = \{n \in \mathbb{Z}: \hat{h}(n) \neq 0\}$ and denote by N_h the subgroup of $\mathbb{Z}$ generated by $\{n_1 - n_2: n_1, n_2 \in S_h\}$.

PROPOSITION 3.2. *Let f be a standard unitary function on $\mathbb{T}$ with $N_f = p\mathbb{Z}$ $(p \in \mathbb{N})$. Then, for $0 \leqslant u < 2\pi$, $\psi_f(e^{iu}) = 1$ if and only if $e^{ipu} = 1$.*

Proof. To prove the sufficiency, assume that $e^{ipu} = 1$ for some $0 \leqslant u < 2\pi$. Let k be any element of S_f. For each $n \in \mathbb{Z}$, the nth Fourier coefficient of the function $e^{it} \to f(e^{i(t+u)}) \overline{f(e^{it})}$ $(0 \leqslant t < 2\pi)$ equals

$$\sum_{m \in S_f} \hat{f}(m) \overline{\hat{f}(m-n)} \, e^{imu} = e^{iku} \sum_{m \in S_f} \hat{f}(m) \overline{\hat{f}(m-n)}.$$

Hence, for each $0 \leqslant t < 2\pi$,

$$f(e^{i(t+u)}) \overline{f(e^{it})} = e^{iku} \, |f(e^{it})|^2 = e^{iku}$$

and, consequently, $\psi_f(e^{iu}) = 1$.

To prove the necessity, assume that $\psi_f(e^{iu}) = 1$ for some $0 \leqslant u < 2\pi$. Then, for each $0 \leqslant s < 2\pi$,

$$\frac{1}{2\pi} \left| \int_0^{2\pi} f(e^{i(s+t)}) \overline{f(e^{it})} \, \overline{f(e^{i(s+t+u)})} \, f(e^{i(t+u)}) \, dt \right| = 1,$$

and so, in view of the unitarity of the integrand and Parseval's identity,

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i(s+t)})\,\overline{f(e^{it})}\,\overline{f(e^{i(s+t+u)})}\,f(e^{i(t+u)})\,e^{-int}\,dt=0$$

whenever $n\in\mathbb{Z}\setminus\{0\}$. It follows that for each $0\leqslant s<2\pi$, the function

$$e^{it}\to f(e^{i(s+t)})\,\overline{f(e^{it})}\,\overline{f(e^{i(s+t+u)})}\,f(e^{i(t+u)}) \qquad (0\leqslant t<2\pi)$$

is constant; hereafter, we shall denote by $c(s)$ the value of this function. For each $0\leqslant t<2\pi$, set

$$h(e^{it})=f(e^{it})\,\overline{f(e^{i(t+u)})}.$$

Clearly, for each $0\leqslant s<2\pi$ and each $0\leqslant t<2\pi$,

$$h(e^{i(s+t)})=c(s)\,h(e^{it}),$$

and so, for each $n\in\mathbb{N}$ and each $0\leqslant s<2\pi$,

$$e^{ins}\hat{h}(n)=c(s)\,\hat{h}(n).$$

Note that if $n_1, n_2\in S_h$, then, by the last equality, $e^{in_1s}=e^{in_2s}$ for all $0\leqslant s<2\pi$, hence $n_1=n_2$. Correspondingly, there exist an integer k and a complex number d of unit modulus such that for each $0\leqslant t<2\pi$,

$$h(e^{it})=de^{ikt}$$

or equivalently

$$f(e^{i(t+u)})=\bar{d}e^{-ikt}f(e^{it}).$$

Actually, $k=0$; for in the contrary case, for any arbitrarily fixed l in S_f and all integers n, we would have

$$|\hat{f}(l+kn)|=|\hat{f}(l)|,$$

which would contradict the Riemann–Lebesgue lemma. Thus, for any $n\in\mathbb{Z}$,

$$e^{inu}\hat{f}(n)=\bar{d}\hat{f}(n),$$

from which it follows that $e^{in_1u}=e^{in_2u}$ for all $n_1, n_2\in S_f$. Hence, finally, $e^{ipu}=1$.

The proof is complete.

To prove the theorem, for each $\omega\in\Omega$, let U_ω be the unitary representation of G associated with the cocycle A_ω. Since, for each $\omega \in \Omega$ and each $g \in G$,

$$(U_\omega(g)\,1, 1) = \int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma),$$

it follows from the purity of the spectrum of the projection-valued measure associated with each U_ω and from Wiener's theorem that for $\{A_\omega\}$ to be non-trivial it is necessary and sufficient that

$$M_g\left(\left|\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma)\right|^2\right) = 0$$

for $\mathbb{P}$-almost all ω in Ω. Letting $\mathbb{E}$ denote the expectation operator relative to $\mathbb{P}$, the latter condition is equivalent to

$$\mathbb{E}_\omega\left[M_g\left(\left|\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma)\right|^2\right)\right] = 0$$

and this in turn, in view of (2.1) and Lebesgue's dominated convergence theorem, is equivalent to

$$M_g\left(\mathbb{E}_\omega\left[\left|\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma)\right|^2\right]\right) = 0. \tag{3.1}$$

By the stochastic independence of $\{\sigma_n : n \in \mathbb{N}\}$ and the fact that each σ_n is uniformly distributed in Σ, for each $g \in G$, we get

$$\begin{aligned}
&\mathbb{E}_\omega\left[\left|\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma)\right|^2\right] \\
&= \mathbb{E}_\omega\left[\int_{\Sigma\times\Sigma} A_\omega(\sigma, g)\,\overline{A_\omega(\sigma', g)}\, dm_\Sigma \otimes m_\Sigma(\sigma, \sigma')\right] \\
&= \int_{\Sigma\times\Sigma} \prod_{n=1}^{\infty} \mathbb{E}_\omega[f_{\chi_n}(\sigma + \sigma_n(\omega))\,\overline{f_{\chi_n}(\sigma' + \sigma_n(\omega))} \\
&\quad \times \overline{f_{\chi_n}(\sigma + \sigma_n(\omega) + \alpha(g))}\, f_{\chi_n}(\sigma' + \sigma_n(\omega) + \alpha(g))]\, dm_\Sigma \otimes m_\Sigma(\sigma, \sigma') \\
&= \int_\Sigma \prod_{n=1}^{\infty} \left(\int_\Sigma f_{\chi_n}(\sigma + \tau)\,\overline{f_{\chi_n}(\tau)}\,\overline{f_{\chi_n}(\sigma + \tau + \alpha(g))}\right. \\
&\quad \left.\times f_{\chi_n}(\tau + \alpha(g))\, dm_\Sigma(\tau)\right) dm_\Sigma(\sigma).
\end{aligned} \tag{3.2}$$

Hence, by the Cauchy–Schwarz inequality,

$$M_g\left(\mathbb{E}_\omega\left[\left|\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma)\right|^2\right]\right)$$
$$\leqslant \left\{M_g\left(\int_\Sigma \prod_{n=1}^{\infty}\left|\int_\Sigma f_{\chi_n}(\sigma+\tau)\,\overline{f_{\chi_n}(\tau)}\,\overline{f_{\chi_n}(\sigma+\tau+\alpha(g))}\right.\right.\right.$$
$$\left.\left.\left.\times f_{\chi_n}(\tau+\alpha(g))\, dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)\right)\right\}^{1/2}. \tag{3.3}$$

Because f has at least two non-zero Fourier coefficients, N_f is a non-zero subgroup of $\mathbb{Z}$. By Proposition 3.2, there is only a finite number of u with $0 \leqslant u < 2\pi$ for which $\psi_f(e^{iu}) = 1$. Therefore $\hat{\psi}_f(0) < 1$. It is clear that given $\varepsilon > 0$, one can find a trigonometric polynomial p on $\mathbb{T}$ and a positive integer k such that

$$(\hat{\psi}_p(0))^k < \varepsilon^2/4 \tag{3.4}$$

and

$$\left\{M_g\left(\int_\Sigma \prod_{i=1}^{k}\left|\int_\Sigma f_{\eta_i}(\sigma+\tau)\,\overline{f_{\eta_i}(\tau)}\,\overline{f_{\eta_i}(\sigma+\tau+\alpha(g))}\right.\right.\right.$$
$$\left.\left.\left.\times f_{\eta_i}(\tau+\alpha(g))\, dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)\right)\right\}^{1/2}$$
$$\leqslant \left\{M_g\left(\int_\Sigma \prod_{i=1}^{k}\left|\int_\Sigma p_{\eta_i}(\sigma+\tau)\,\overline{p_{\eta_i}(\tau)}\,\overline{p_{\eta_i}(\sigma+\tau+\alpha(g))}\right.\right.\right.$$
$$\left.\left.\left.\times p_{\eta_i}(\tau+\alpha(g))\, dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)\right)\right\}^{1/2} + \frac{\varepsilon}{2} \tag{3.5}$$

for every $\eta_1, ..., \eta_k \in \hat{\Sigma}$.

Let $N = \max\{|n| : n \in S_p\}$. Direct computation shows that for any $\eta \in \hat{\Sigma}$ of order d,

$$\int_\Sigma p_\eta(\sigma+\tau)\,\overline{p_\eta(\tau)}\,\overline{p_\eta(\sigma+\tau+\alpha(g))}\,p_\eta(\tau+\alpha(g))\, dm_\Sigma(\tau)$$
$$= \begin{cases} \displaystyle\sum_{j-k-l+m \in d\mathbb{Z}} \hat{p}(j)\,\overline{\hat{p}(k)}\,\overline{\hat{p}(l)}\,\hat{p}(m)((j-l)\,\sigma + (m-l)\,\alpha(g), \eta) & \text{if } d < +\infty \\ \displaystyle\sum_{j-k-l+m=0} \hat{p}(j)\,\overline{\hat{p}(k)}\,\overline{\hat{p}(l)}\,\hat{p}(m)((j-l)\,\sigma + (m-l)\,\alpha(g), \eta) & \text{if } d = +\infty; \end{cases}$$

hence, if η has order $>4N$, then

$$\int_\Sigma p_\eta(\sigma+\tau)\,\overline{p_\eta(\tau)}\,\overline{p_\eta(\sigma+\tau+\alpha(g))}\,p_\eta(\tau+\alpha(g))\,dm_\Sigma(\tau)$$

$$= \sum_{j-k-l+m=0} \hat{p}(j)\,\overline{\hat{p}(k)}\,\overline{\hat{p}(l)}\,\hat{p}(m)((j-l)\,\sigma+(m-l)\,\alpha(g),\eta). \tag{3.6}$$

We select now an increasing sequence $(n_i)_{1\leqslant i\leqslant k}$ of positive integers so that the following condition is satisfied:

> given integers d_i of modulus $\leqslant 4N$ $(1\leqslant i\leqslant k)$, if $\sum_{i=0}^k d_i\chi_{n_i}=0$, then $d_i=0$ for each i. $(*)$

To this end, we first pick $n_1\in\mathbb{N}$ so that $\operatorname{ord}\chi_{n_1}>8N$ and then find a neighbourhood U of 0 in $\hat{\Sigma}$ such that if d_1 runs over the integers of modulus $\leqslant 4N$, then the sets $d_1\chi_{n_1}+U$ are pairwise disjoint. We next choose an integer $n_2>n_1$ so that $\operatorname{ord}\chi_{n_2}>8N$ and, for each integer d_2 of modulus $\leqslant 4N$, $d_2\chi_{n_2}$ is in U; at this stage, the $d_1\chi_{n_1}+d_2\chi_{n_2}$ are all distinct. Now it is clear how to continue the process to find the remaining terms of $(n_i)_{1\leqslant i\leqslant k}$.

Taking into account that each χ_{n_i} $(1\leqslant i\leqslant k)$ has order $>4N$, we can now apply (3.6) and $(*)$ to get

$$\int_\Sigma \prod_{i=1}^k \left|\int_\Sigma p_{\chi_{n_i}}(\sigma+\tau)\,\overline{p_{\chi_{n_i}}(\tau)}\,\overline{p_{\chi_{n_i}}(\sigma+\tau+\alpha(g))}\right.$$
$$\left.\times\, p_{\chi_{n_i}}(\tau+\alpha(g))\,dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)$$
$$=\Sigma'\prod_{i=1}^k \hat{p}(j_i)\,\overline{\hat{p}(j_i')}\,\overline{\hat{p}(k_i)}\,\hat{p}(k_i')\,\overline{\hat{p}(l_i)}\,\hat{p}(l_i')\,\hat{p}(m_i)\,\overline{\hat{p}(m_i')}$$
$$\times\,((m_i-m_i'-l_i+l_i')\,\alpha(g),\chi_{n_i}), \tag{3.7}$$

where the dashed sum extends over the 8-tuples of integers $(j_i, j_i', k_i, k_i', l_i, l_i', m_i, m_i')$ with $j_i-k_i-l_i+m_i=j_i'-k_i'-l_i'+m_i'=j_i-j_i'-l_i+l_i'=0$. Making appeal to $(*)$ once again, we obtain furthermore that

$$M_g\left(\int_\Sigma \prod_{i=1}^k \left|\int_\Sigma p_{\chi_{n_i}}(\sigma+\tau)\,\overline{p_{\chi_{n_i}}(\tau)}\,\overline{p_{\chi_{n_i}}(\sigma+\tau+\alpha(g))}\right.\right.$$
$$\left.\left.\times\, p_{\chi_{n_i}}(\tau+\alpha(g))\,dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)\right)$$
$$=\Sigma''\prod_{i=1}^k \hat{p}(j_i)\,\overline{\hat{p}(j_i')}\,\overline{\hat{p}(k_i)}\,\hat{p}(k_i')\,\overline{\hat{p}(l_i)}\,\hat{p}(l_i')\,\hat{p}(m_i)\,\overline{\hat{p}(m_i')},$$

where this time the dashed sum extends over the 8-tuples $(j_i, j_i', k_i, k_i', l_i, l_i', m_i, m_i')$ with $j_i - k_i - l_i + m_i = j_i' - k_i' - l_i' + m_i' = j_i - j_i' - l_i + l_i' = m_i - m_i' - l_i + l_i' = 0$. It is readily verified that the right-hand side of the latter identity equals $(\hat{\psi}_p(0))^k$. Thus, in view of (3.3), (3.4), and (3.5),

$$M_g\left(\mathbb{E}_\omega\left[\left|\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma)\right|^2\right]\right)$$
$$\leqslant \left\{M_g\left(\int_\Sigma \prod_{i=1}^{k} \left|\int_\Sigma f_{\chi_{n_i}}(\sigma+\tau)\overline{f_{\chi_{n_i}}(\tau)}\,\overline{f_{\chi_{n_i}}(\sigma+\tau+\alpha(g))}\right.\right.\right.$$
$$\left.\left.\left.\times f_{\chi_{n_i}}(\tau+\alpha(g))\, dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)\right)\right\}^{1/2} < \varepsilon.$$

By the arbitrariness of ε, equality (3.1) follows.

The proof is complete.

4. Random Cocycles of Oscillatory Type

In this section we show that for each standard unitary function f on $\mathbb{T}$ with absolutely convergent Fourier series, each sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive numbers with $\sum_{n=1}^{\infty} \omega_f(\delta_n) < +\infty$, each standard sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G, and each function i mapping $\mathbb{N}$ into itself with $\lim_{n\to\infty} i(n) = +\infty$, there exists a sequence $(\chi_n)_{n\in\mathbb{N}}$ in $\hat{\Sigma}$ satisfying conditions (i) and (ii) of the preceding section, such that the corresponding random cocycle is of oscillatory and, a fortiori, of singular continuous type.

Let f be a standard unitary function on $\mathbb{T}$ with absolutely convergent Fourier series. Since

$$\sum_{m=0}^{\infty} \sum_{|k-l|\geqslant m} |\hat{f}(k)\hat{f}(l)| \leqslant \left(\sum_{n\in\mathbb{Z}} |\hat{f}(n)|\right)^2,$$

there exists $p \in \mathbb{N}$ such that

$$\sum_{m=p+1}^{\infty} \sum_{|k-l|\geqslant m} |\hat{f}(k)\hat{f}(l)| < \tfrac{1}{4}. \tag{4.1}$$

Given $0 \leqslant s < 2\pi$, put

$$\varphi(e^{is}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{f(e^{i(s+t)})}\, dt.$$

For each $n \in \mathbb{N}$, let β_n be a positive number such that

$$|e^{is} - 1| < \beta_n \qquad (0 \leqslant s < 2\pi)$$

implies

$$|\varphi(e^{is})-1|<\frac{1}{(n+1)^2}.$$

Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty}\omega_f(\delta_n)<+\infty$, $(K_n)_{n\in\mathbb{N}}$ be a standard sequence of compact subsets of G, and i be a function from $\mathbb{N}$ into itself such that $\lim_{n\to\infty} i(n)=+\infty$. We shall inductively define two sequences, $(\chi_n)_{n\in\mathbb{N}}$ in $\hat{\Sigma}$ and $(g_n)_{n\in\mathbb{N}}$ in G, such that

(α) ord $\chi_n>p+n-1$ for each $n\in\mathbb{N}$;

(β) for each $n\in\mathbb{N}$, if $g\in K_{i(n)}$, then

$$|(\alpha(g),\chi_n)-1|<\delta_n;$$

(γ) $\lim_{n\to\infty} g_n=\infty$;

(δ) $|(\alpha(g_k),\chi_n)-1|<\beta_n$ for all $k,n\in\mathbb{N}$.

Note that the sequence $(\chi_n)_{n\in\mathbb{N}}$ so defined will satisfy conditions (i) and (ii) of the foregoing section.

In the first step, we pick g_1 in $G\setminus K_1$ and next choose χ_1 in $\hat{\Sigma}$ so that ord $\chi_1>p$,

$$|(\alpha(g),\chi_1)-1|<\delta_1$$

for each $g\in K_{i(1)}$, and

$$|(\alpha(g_1),\chi_1)-1|<\beta_1.$$

Suppose we have chosen $(\chi_j)_{1\leqslant j\leqslant n}$ and $(g_j)_{1\leqslant j\leqslant n}$. In the $(n+1)$st step, we select χ_{n+1} in $\hat{\Sigma}$ so that ord $\chi_{n+1}>p+n$,

$$|(\alpha(g),\chi_{n+1})-1|<\delta_{n+1}$$

for each $g\in K_{i(n+1)}$, and

$$|(\alpha(g_j),\chi_{n+1})-1|<\beta_{n+1}$$

for $1\leqslant j\leqslant n$. Next we choose g_{n+1} in $G\setminus K_{n+1}$ so that

$$|(\alpha(g_{n+1}),\chi_j)-1|<\beta_j$$

for $1\leqslant j\leqslant n+1$ by making use of the following remark: Given any compact subset K of G, the set $\alpha(G\setminus K)$ is dense in Σ; in particular, $\alpha(G\setminus K)$ intersects each neighbourhood of 0 in Σ. The truth of the remark is seen as follows. Suppose, contrariwise, that $\alpha(G\setminus K)$ is not dense in Σ. Then there exists a non-negative non-zero continuous function f on Σ vanishing on $\alpha(G \setminus K)$. Clearly, $f \circ \alpha$ is an almost periodic function on G. Letting $(H_n)_{n \in \mathbb{N}}$ be a sequence of compact subsets of G such that equality (2.1) holds, we have

$$M(f \circ \alpha) = \lim_{n \to \infty} \frac{1}{m_G(H_n)} \int_K f(\alpha(g))\, dm_G(g) = 0.$$

On the other hand,

$$\int_\Sigma f\, dm_\Sigma = M(f \circ \alpha)$$

(cf. [15, Theorem 26.17]), hence $f = 0$, a contradiction.

We are now in a position to state the following.

THEOREM 4.1. *With f, $(\delta_n)_{n \in \mathbb{N}}$, $(K_n)_{i \in \mathbb{N}}$, i, and $(\chi_n)_{n \in \mathbb{N}}$ as above, the corresponding random cocycle $\{A_\omega\}$ is of oscillatory type.*

Proof. We retain the notation from the preceding section. By the stochastic independence of $\{\sigma_n : n \in \mathbb{N}\}$ and the fact that each σ_n is uniformly distributed in Σ, for each $g \in G$, we have

$$\begin{aligned}
&\mathbb{E}_\omega \left[\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma) \right] \\
&= \int_\Sigma \prod_{n=1}^{\infty} \mathbb{E}_\omega [f_{\chi_n}(\sigma + \sigma_n(\omega)) \overline{f_{\chi_n}(\sigma + \sigma_n(\omega) + \alpha(g))}]\, dm_\Sigma(\sigma) \\
&= \prod_{n=1}^{\infty} \int_\Sigma f_{\chi_n}(\sigma) \overline{f_{\chi_n}(\sigma + \alpha(g))}\, dm_\Sigma(\sigma). \qquad (4.2)
\end{aligned}$$

It is easy to verify that for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\int_\Sigma f_{\chi_n}(\sigma) \overline{f_{\chi_n}(\sigma + \alpha(g))}\, dm_\Sigma(\sigma) \\
&\qquad = \varphi((\alpha(g), \chi_n)) + \Sigma' \hat{f}(k) \overline{\hat{f}(l)} (-l\alpha(g), \chi_n),
\end{aligned}$$

where the dashed sum extends over all integers k, l such that $|k - l|$ is a non-zero multiple of ord χ_n. The last equality combined with (α), (4.1), and (4.2) shows that for each $g \in G$,

$$\left| \mathbb{E}_\omega \left[\int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma) \right] - \prod_{n=1}^{\infty} \varphi((\alpha(g), \chi_n)) \right| < \tfrac{1}{4}. \qquad (4.3)$$

By (δ), for all $k, n \in \mathbb{N}$,

$$|\varphi((\alpha(g_k), \chi_n)) - 1| < \frac{1}{(k+1)^2}.$$

Hence, for each $k \in \mathbb{N}$,

$$\left|\prod_{n=1}^{\infty} \varphi((\alpha(g_k), \chi_n))\right| > \prod_{n=1}^{\infty} \left(1 - \frac{1}{(n+1)^2}\right) = \frac{1}{2}.$$

This together with (4.3) shows that for each $k \in \mathbb{N}$,

$$\left|\mathbb{E}_\omega\left[\int_\omega A_\omega(\sigma, g_k)\, dm_\Sigma(\sigma)\right]\right| > \tfrac{1}{4}. \tag{4.4}$$

If $\{A_\omega\}$ were not of oscillatory type, then, by (γ) and Theorem 2.1, we would have

$$\lim_{k \to \infty} \int_\Sigma A_\omega(\sigma, g_k)\, dm_\Sigma(\sigma) = 0$$

for $\mathbb{P}$-almost all ω in Ω, from which, by Lebesgue's dominated convergence theorem, it would follow that

$$\lim_{k \to \infty} \mathbb{E}_\omega\left[\int_\Sigma A_\omega(\sigma, g_k)\, dm_\Sigma(\sigma)\right] = 0,$$

contrary to (4.4).

This completes the proof.

5. Random Cocycles of Haar Type

In this section we show that for each standard unitary function f on $\mathbb{T}$ with $N_f = \mathbb{Z}$, there exist a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive numbers with $\sum_{n=1}^{\infty} \omega_f(\delta_n) < +\infty$, a standard sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets of G, a function i mapping $\mathbb{N}$ into itself with $\lim_{n \to \infty} i(n) = +\infty$, and a sequence $(\chi_n)_{n \in \mathbb{N}}$ in $\hat{\Sigma}$ satisfying conditions (i) and (ii) of Section 3, such that the corresponding random cocycle is of Haar type.

We start with some auxiliary results.

Proposition 5.1. *Let $\{\gamma_i : i = 1, ..., k\}$ be a finite independent subset of $\hat{G}$ and f be a complex continuous function on $\mathbb{T}$. Then*

$$\lim_{\substack{\hat{\alpha}(\eta_1)\to\gamma_1\\ \cdots\\ \hat{\alpha}(\eta_k)\to\gamma_k}} \int_\Sigma \prod_{i=1}^{k} \left| \int_\Sigma f_{\eta_i}(\sigma+\tau)\overline{f_{\eta_i}(\tau)}\,\overline{f_{\eta_i}(\sigma+\tau+\alpha(g))} \times f_{\eta_i}(\tau+\alpha(g))\,dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)$$

$$= \prod_{i=1}^{k} \psi_f((g,\gamma_i)) \qquad (\eta_i\in\hat{\Sigma}, g\in G) \tag{5.1}$$

with locally uniform convergence in g.

Proof. Since both sides of (5.1) vary continuously as f is varied continuously in the supremum norm, without loss of generality we may assume that f is a polynomial on $\mathbb{T}$. Let $N=\max\{|n|: n\in S_f\}$. By the independency of $\{\gamma_i: i=1, ..., k\}$, there exists a neighbourhood U of 0 in $\hat{G}$ such that given $\eta_i\in\hat{\Sigma}$ $(1\leqslant i\leqslant k)$, if $\hat{\alpha}(\eta_i)\in\gamma_i+U$ for each i, then condition $(*)$ of the proof of Theorem 3.1 is fulfilled. Suppose from now on that $\hat{\alpha}(\eta_i)\in\gamma_i+U$ for each i. Then, in analogy to Eq. (3.7), one has

$$\int_\Sigma \prod_{i=1}^{k} \left| \int_\Sigma f_{\eta_i}(\sigma+\tau)\overline{f_{\eta_i}(\tau)}\,\overline{f_{\eta_i}(\sigma+\tau+\alpha(g))} \times f_{\eta_i}(\tau+\alpha(g))\,dm_\Sigma(\tau)\right|^2 dm_\Sigma(\sigma)$$

$$= \Sigma' \prod_{i=1}^{k} \hat{f}(j_i)\overline{\hat{f}(j_i')}\,\overline{\hat{f}(k_i)}\,\hat{f}(k_i')\,\overline{\hat{f}(l_i)}\,\hat{f}(l_i')\,\hat{f}(m_i)\,\overline{\hat{f}(m_i')} \times ((m_i-m_i'-l_i+l_i')\,\alpha(g),\eta_i),$$

where the dashed sum extends over the 8-tuples of integers $(j_i, j_i', k_i, k_i', l_i, l_i', m_i, m_i')$ with $j_i-k_i-l_i+m_i=j_i'-k_i'-l_i'+m_i'=j_i-j_i'-l_i+l_i'=0$. Direct verification shows that the right-hand side of the identity above equals $\prod_{i=1}^{k}\psi_f((\alpha(g),\eta_i))$. To end the proof, it suffices to note that

$$\lim_{\substack{\hat{\alpha}(\eta_1)\to\gamma_1\\ \cdots\\ \hat{\alpha}(\eta_k)\to\gamma_k}} \prod_{i=1}^{k} \psi_f((\alpha(g),\eta_i)) = \prod_{i=1}^{k}\psi_f((g,\gamma_i))$$

with locally uniform convergence in g.

PROPOSITION 5.2. *Let G be a locally compact non-compact Abelian group, K be a compact subset of G, and δ be a positive number. Then there* *exists a compact subset L of G such that for each g in $G\backslash L$, there is ξ in $\hat{G}$ with*

$$|(g, \xi)+1| \leqslant 1$$

and

$$|(k, \xi)-1| < \delta$$

for all k in K.

Proof. Let U be a compact symmetric neighbourhood of 0 in G. K being compact, there exist $a_1, ..., a_n$ in K with

$$K \subset \bigcup_{i=1}^{n} (a_i + U).$$

Let

$$V = \bigcup_{i=1}^{n} (-a_i + U) \cup (a_i + U).$$

Clearly, V is a compact symmetric neighbourhood of 0 in G containing K. It is immediate that

$$G_0 = \bigcup_{n=1}^{\infty} nV$$

is an open subgroup of G. By the principal structure theorem (cf. [19, Theorem 24]), there exist non-negative integers k and l and a compact Abelian group H such that G_0 is topologically isomorphic to $\mathbb{R}^k \oplus \mathbb{Z}^l \oplus H$. Denote by β the canonical homomorphism from G_0 onto $\mathbb{R}^k \oplus \mathbb{Z}^l$. We shall consider two cases:

(i) $G_0 \neq H$. We write $\cdot$ for the Euclidean scalar product and $\|\cdot\|$ for the Euclidean norm on Euclidean space. Assume that $\mathbb{Z}^k$ is canonically embedded in $\mathbb{R}^k$. Let r be a positive number such that for all $(x, y) \in \beta(K)$ and all $(s, t) \in \mathbb{R}^k \times \mathbb{R}^l$ with $\|(s, t)\| < r$,

$$|\exp(i(x \cdot s + y \cdot t)) - 1| < \delta.$$

Let

$$L = \{(x, y, h) \in \mathbb{R}^k \oplus \mathbb{Z}^l \oplus H \colon \|(x, y)\| \leqslant \pi/r\}.$$

Given any (x', y', h') in $G_0 \backslash L$, we can find a continuous character ξ of G whose restriction on G_0 takes the form

$$((x, y, h), \xi) = \exp\left(\frac{\pi i(x \cdot x' + y \cdot y')}{\|(x', y')\|^2}\right) \qquad ((x, y, h) \in G_0).$$

Clearly

$$((x', y', h'), \xi) = -1$$

and

$$|(k, \xi) - 1| < \delta$$

whenever $k \in K$. Thus when $G = G_0$, the proof is already finished. When $G \neq G_0$, for any given g in $G \setminus G_0$, we can find ξ in $\hat{G}$ such that $(g, \xi) \neq 1$ and $(g', \xi) = 1$ for all $g' \in G_0$ (cf. [15, Corollary 23.26]). By passing, if necessary, to a suitable multiple of ξ, we get $|(g, \xi) + 1| \leqslant 1$. This completes the proof in the other case.

(ii) $G_0 = H$. By the argument of the preceding paragraph, given any g in $G \setminus G_0$, there is ξ in $\hat{G}$ such that $|(g, \xi) + 1| \leqslant 1$ and $(g', \xi) = 1$ for all $g' \in G_0$. To complete the proof, it suffices to take G_0 for L.

Let V be a compact neighbourhood of 0 in G. Since G is σ-compact, there is a subset $\{a_n : n \in \mathbb{N}\}$ of G such that

$$G = \bigcup_{n=1}^{\infty} (a_n + V).$$

Of course, since G is not compact, we may assume that $\lim_{n \to \infty} a_n = \infty$. For each $n \in \mathbb{N}$, let

$$K_n = \bigcup_{k=1}^{n} (a_k + V).$$

Clearly, $(K_n)_{n \in \mathbb{N}}$ is a standard sequence of compact subsets of G.

Let f be a standard unitary function on $\mathbb{T}$ with $N_f = \mathbb{Z}$ and $(\beta_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} n\omega_f(\beta_n) < +\infty$.

In view of Proposition 5.2 and the fact that $\lim_{n \to \infty} a_n = \infty$, given any $j \in \mathbb{N}$, there exists $m_j \in \mathbb{N}$ such that for each integer $n \geqslant m_j$, there is $\xi_{j,n}$ in $\hat{G}$ with

$$|(a_n, \xi_{j,n}) + 1| \leqslant 1$$

and

$$|(g, \xi_{j,n}) - 1| < \beta_n$$

for all g in K_j. It is clear that for each $n \in \mathbb{N}$, there exists exactly one $j(n) \in \mathbb{N}$ such that

$$m_1 + \cdots + m_{j(n)} \leqslant n + m_1 - 1 < m_1 + \cdots + m_{j(n)+1}.$$

Given $n \in \mathbb{N}$, set

$$\xi_n = \xi_{j(n), n+m_1-1};$$

then

$$|(a_{n+m_1-1}, \xi_n) + 1| \leqslant 1$$

and

$$|(g, \xi_n) - 1| < \beta_{n+m_1-1}$$

for all g in $K_{j(n)}$.

$\hat{G}$ is a non-discrete I-group, so given ξ in $\hat{G}$, a neighbourhood U of 0 in $\hat{G}$, and a positive integer p, there exists an independent subset of $\xi + U$ with p elements (cf. [24, Lemma 5.2.3]). Thus, by the foregoing, we can find a sequence $(\gamma_n)_{n \in \mathbb{N}}$ in $\hat{G}$ such that for each $n \in \mathbb{N}$, $\{\gamma_k : n^2 \leqslant k < (n+1)^2\}$ is an independent set, and if $n^2 \leqslant k < (n+1)^2$, then

$$|(a_{n+m_1-1}, \gamma_k) + 1| < \tfrac{3}{2}$$

and

$$|(g, \gamma_k) - 1| < \beta_{n+m_1-1}$$

for all g in $K_{j(n)}$.

Since $N_f = \mathbb{Z}$, it follows from Proposition 3.2 that there exists $0 < \delta < 1$ such that for each $0 \leqslant u < 2\pi$ with $|e^{iu} + 1| < \frac{3}{2}$, $|\psi_f(e^{iu})| < \delta$. Thus, in view of the preceding paragraph and Proposition 5.1, we can find a sequence $(\chi_n)_{n \in \mathbb{N}}$ in $\hat{\Sigma}$ such that for each positive integer n, if $n^2 \leqslant k < (n+1)^2$, then ord $\chi_k > n$,

$$|(\alpha(g), \chi_k) - 1| < \beta_{n+m_1-1}$$

for all g in $K_{j(n)}$, and

$$\begin{aligned}\int_\Sigma \prod_{k=n^2}^{n^2+2n} \bigg| \int_\Sigma f_{\chi_k}(\sigma+\tau)\overline{f_{\chi_k}(\tau)}\,\overline{f_{\chi_k}(\sigma+\tau+\alpha(g))} \\ \times f_{\chi_k}(\tau+\alpha(g))\, dm_\Sigma(\tau)\bigg|^2 dm_\Sigma(\sigma) \\ < \prod_{k=n^2}^{n^2+2n} \psi_f((g, \gamma_k)) + \delta^{2n+1}\end{aligned}$$

for all g in $a_{n+m_1-1} + V$. Since the sequence $(\gamma_n)_{n \in \mathbb{N}}$ tends to 0 in $\hat{G}$ as $n \to \infty$, there exists an integer $n_0 \geqslant 2$ such that for each integer $n \geqslant n_0$, if

$n^2 \leqslant k < (n+1)^2$ and $g \in a_{n+m_1-1} + V$, then $|(g, \gamma_k) + 1| < \frac{3}{2}$. Consequently, for each integer $n \geqslant n_0$ and each g in $a_{n+m_1-1} + V$,

$$\int_\Sigma \prod_{k=n^2}^{n^2+2n} \left| \int_\Sigma f_{\chi_k}(\sigma+\tau) \overline{f_{\chi_k}(\sigma)} \overline{f_{\chi_k}(\sigma+\tau+\alpha(g))} \right.$$

$$\left. \times f_{\chi_k}(\tau+\alpha(g))\, dm_\Sigma(\tau) \right|^2 dm_\Sigma(\sigma) < 2\delta^{2n+1}. \tag{5.2}$$

Given $n \in \mathbb{N}$, set

$$\delta_n = \beta_{[n^{1/2}]+m_1-1}, \qquad i(n) = j([n^{1/2}]),$$

where $[n^{1/2}]$ denotes the integral part of $n^{1/2}$. It is immediately clear that $\lim_{n\to\infty} i(n) = +\infty$, $\sum_{n=1}^\infty \omega_f(\delta_n) < +\infty$, and that the sequence $(\chi_n)_{n\in\mathbb{N}}$ satisfies conditions (i) and (ii) of Section 3.

We can now state the main conclusion of this section.

THEOREM 5.3. *With f, $(\delta_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$, i, and $(\chi_n)_{n\in\mathbb{N}}$ as above, the corresponding random cocycle $\{A_\omega\}$ is of Haar type.*

Proof. We retain the notation from the foregoing sections. It follows from (3.2), the Cauchy–Schwarz inequality, and Fibini's theorem that

$$\mathbb{E}_\omega \left[\int_G \left| \int_\Sigma A_\omega(\sigma, g)\, dm_\Sigma(\sigma) \right|^2 dm_G(g) \right]$$

$$\leqslant \int_G \left(\int_\Sigma \prod_{n=1}^\infty |f_{\chi_n}(\sigma+\tau) \overline{f_{\chi_n}(\tau)} \overline{f_{\chi_n}(\sigma+\tau+\alpha(g))} \right.$$

$$\left. \times f_{\chi_n}(\tau+\alpha(g))\, dm_\Sigma(\tau)|^2\, dm_\Sigma(\sigma) \right)^{1/2} dm_G(g).$$

By (5.2), the right-hand side of this inequality does not exceed

$$m_G\left(\bigcup_{n=1}^{n_0+m_1-2} K_n \right) + 2^{1/2}\delta^{n_0+1/2}(1-\delta)^{-1}\, m_G(V).$$

Therefore, for $\mathbb{P}$-almost all ω in Ω, $g \to (U_\omega g)\, 1, 1)$ is a square Haar-integrable function on G, which, by Plancherel's theorem, is the Fourier transform of a square Haar-integrable function on $\hat{G}$. Since for each ω in Ω, $g \to (U_\omega(g)\, 1, 1)$ is the Fourier transform of a measure in $M(\hat{G})$, it follows that for $\mathbb{P}$-almost all ω in Ω, $g \to (U_\omega(g)\, 1, 1)$ is the Fourier transform of a Haar-integrable function on $\hat{G}$ (cf. [15, Theorem 31.33]). Hence, by the purity of the spectrum of the projection-valued measure associated with each U_ω, we finally infer that the random cocycle $\{A_\omega\}$ is of Haar type.

The proof is complete.

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