Spectral analysis of Schrödinger operators in non-separable Hilbert spaces

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SPECTRAL ANALYSIS OF SCHRÖDINGER OPERATORS IN NON-SEPARABLE HILBERT SPACES

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1. *Introduction*

Any physical system with n degrees of freedom can be described in terms of n pais q_i, p_i of canonical variables, where $q_1, \ldots, q_n$ are the so-called position variables and $p_1, \ldots, p_n$ the so-called momentum variables of the system. From the classical point of view the state of the system is determined by the results of measuring the q_i and p_i at a given instant. However, as subtle experiments show, the measurements of the q_i and p_i performed repeatedly in a short period of time provide statistical data such that each product $\sigma_{q_i} \cdot \sigma_{p_i}$ of the corresponding dispersions σ_{q_i}, σ_{p_i} is not less than a small but non-zero universal constant. The impossibility of effective experimental determination of the simultaneous positions and momenta of the system necessitates a suitable modification of the notion of state. It turns out that a physically reasonable modification is obtained by identifying the state of the system with an ample collection of the experimental mean values of functions of the canonical variables. Quantum mechanics sets up a correspondence between this phenomenological notion of state and some mathematical concepts that appears within the framework of the so-called formalism of non-commuting canonical variables. We shall discuss this identification in more detail.

There is a unique, up to *-isomorphism, C*-algebra A_n with unit e which is generated, as a C*-algebra, by two subsets $\{u(t): t \in R^n\}$, $\{v(t): t \in R^n\}$ such that

(1) $u(0) = v(0) = e$,

(2) $u(s)u(t) = u(s+t)$ for $s, t \in R^n$,

(3) $v(s)v(t) = v(s+t)$ for $s, t \in R^n$,

This paper is in final form and no version of it will be submitted for publication elsewhere.

(4) $u^*(t) = u(-t)$ for $t \in R^n$,

(5) $v^*(t) = v(-t)$ for $t \in R^n$,

(6) $u(s)v(t) = e^{is\cdot t}v(t)u(s)$ for $s, t \in R^n$, where $s\cdot t=s_1t_1+\ldots+s_nt_n$

(cf.[1], Th.5.2.8). A_n is called the canonical commutation relations algebra over R^n. A state ω over A_n is a linear bounded functional such that $||\omega|| = 1 = \omega(e)$. Let E_{A_n} be the set of all states over A_n. E_{A_n} is convex and, under the *-weak topology, it is compact. By the Krein-Milman theorem, the set $E(E_{A_n})$ of all extreme points of E_{A_n} is not empty. Any element of $E(E_{A_n})$ is called a pure state over A_n. Via the GNS construction (cf.[1], Th. 2.3.16), with each $\omega \in E_{A_n}$ one can associate an essentially unique triple $(H_\omega, \Omega_\omega, \pi_\omega)$, where H_ω is a Hilbert space, Ω_ω is an element of H_ω, and π_ω is a *-representation of A_n in H_ω such that

(1) the space $\{\pi_\omega(A)\Omega_\omega : A \in A_n\}$ is dense in H_ω (cyclicity of Ω_ω and π_ω),

(2) $\omega(B^*A) = (\pi_\omega(A)\ \ , \pi_\omega(B)\Omega_\omega)$ for $A, B \in A_n$.

If ω is a pure state over A_n, then the corresponding cyclic representation π_ω is irreducible. A state ω over A_n is called regular if the representations $t \to \pi_\omega(u(t))$ and $t \to \pi_\omega(v(t))$ of R^n in H_ω are strongly continuous. We let $r(E_{A_n})$ denote the set of all regular states over A_n.

Quantum mechanics is based on the postulate that each phenomenological state of a system with n degrees of freedom can be identified with an element ω of $E(E_{A_n}) \cap r(E_{A_n})$ in such a way that, in suitable units, for all $s, t \in R^n$, the experimental mean value of $e^{is\cdot p}e^{it\cdot q}$

$$= \frac{1}{2}\,\omega(u(s)v(t) + v(t)u(s)),$$

where $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$. The assumption that ω is not a mixture of distinct states over A_n reflects the fact that the phenomenological state is supposed to be determined with maximal accuracy. The regularity assumption is of technical character and is introduced to make it possible to define the quantum mechanical positions $Q_i^{(\pi)}$ and momenta $P_i^{(\pi)}$ (non-commuting canonical variables) as the products of $1/\sqrt{-1}$ and the infinitesimal generators of the groups $t \to \pi_\omega(v_i(t))$ and $t \to \pi_\omega(u_i(t))$, respectively, where

$$v_i(t) = v((\underbrace{0, \ldots, 0}_{i-1 \text{ entries}}, t, 0, \ldots, 0)),$$

$$u_i(t) = u((\underbrace{0, \ldots, 0}_{i-1 \text{ entries}}, t, 0, \ldots, 0)).$$

That this definition of the $Q_i^{(\pi)}$ and $P_i^{(\pi)}$ does not depend on ω is a consequence of the Stone-von Neumann theorem stating that for any $\omega, \omega' \in E(E_{A_n}) \cap r(E_{A_n})$ one can identify H_ω with $H_{\omega'}$ and π_ω with $\pi_{\omega'}$ (cf.[1], Cor.5.2.15). The essentially unique cyclic representation π of A_n associated with the pure regular states over A_n is called the regular irreducible representation of A_n. It can be realized in $L^2(R^n)$ as the unique representation of A_n satisfying

$$\pi(u(t))f = T_t f,$$

$$(t \in R^n,\ f \in L^2(R^n))$$

$$\pi(v(t))f = e_t f$$

where $T_t f(x) = f(x+t)$ and $e_t(x) = e^{it\cdot x}$ for all $x \in R^n$. The necessity of introducing the $Q_i^{(\pi)}$ and $P_i^{(\pi)}$ arises naturally when one wants to describe time evolution of the system. The evolution is governed by a strongly continuous one parameter group $(G_t)_{t\in R}$ of unitary operators on $L^2(R^n)$ in such a way that if $\Omega \in L^2(R^n)$ corresponds to an initial state of the system, then $G_t\Omega$ corresponds to the state after time t. The infinitesimal generator of $(G_t)_{t\in R}$ times $1/\sqrt{-1}$, called the Schrödinger operator or Hamiltonian of the system, completely determines $(G_t)_{t\in R}$. It takes the form

$$H_\pi = -[(P_1^{(\pi)})^2 + \ldots + (P_n^{(\pi)})^2] + V_\pi$$

where $V_\pi = V(Q_1^{(\pi)}, \ldots, Q_n^{(\pi)})$ and V, called a potential, is a real-valued Borel function on R^n containing information about all the peculiarities of the system.

If V is a special potential of the form

$$V = \sum_{\lambda\in R^n} a_\lambda e_\lambda,$$

where $\overline{a_\lambda} = a_{-\lambda} \in C$ for each λ and the sum extends only over finite number of non-zero λ, then the corresponding operator V_π can directly be defined in terms of π as being

$$\sum_{\lambda \in R^n} a_\lambda \pi(v_1(\lambda_1) \cdot \ldots \cdot v_n(\lambda_n)).$$

The latter sum makes sense if π is replaced by any irreducible representation ρ of A_n. In particular, if ρ is such that the unitary groups $t \to \rho(u_i(t))$ are strongly continuous and, correspondingly, have infinitesimal generators $P_i^{(\rho)}$, then one can define a generalized Hamiltonian

$$H_\rho = -[(P_1^{(\rho)})^2 + \ldots + (P_n^{(\rho)})^2] + V_\rho,$$

where

$$V_\rho = \sum_{\lambda \in R^n} a_\lambda \rho(v_1(\lambda_1) \cdot \ldots \cdot v_n(\lambda_n)).$$

More generally, if there is a net $((a_\lambda^{(\alpha)})_{\lambda \in R^n})_{\alpha \in A}$ in C^{R^n} such that $\overline{a_\lambda^{(\alpha)}} = a_{-\lambda}^{(\alpha)}$ for each α and each λ, $a_\lambda^{(\alpha)} \neq 0$ only for finitely many λ when α is fixed, and the limits

$$\lim_\alpha \sum_{\lambda \in R^n} a_\lambda^{(\alpha)} \pi(v_1(\lambda_1) \cdot \ldots \cdot v_n(\lambda_n)),$$

$$\lim_\alpha \sum_{\lambda \in R^n} a_\lambda^{(\alpha)} \rho(v_1(\lambda_1) \cdot \ldots \cdot v_n(\lambda_n))$$

exist in the sense of strong convergence, then the operators V_π and V_ρ defined by these limits give rise to two Hamiltonians H_π and H_ρ. While H_ρ seems to have no direct physical interpretation, it is conceivable that there may exist a mathematically interesting relationship between H_π and H_ρ. In the sequel, we shall provide a few examples that will substantiate this supposition. First, however, we shall take a closer look at irreducible representations of A_n.

2. *Irreducible representations of* A_n.

Among representations of A_n the most interesting from our point of view are semiregular ones. These are such non-regular representations of ρ of A_n for which each group $t \to \rho(u_i(t))$ is strongly continuous. Any representation of A_n that is neither regular nor semiregular will be called irregular.

The basic example of a semiregular irreducible representation of A_n involves the Bohr compactification bR^n of R^n. bR^n is a compact Abelian group with the property that there is a continuous injective homomorphism $\alpha: R^n \to bR^n$

such that $\alpha(R^n)$ is dense in bR^n and, for each $t \in R^n$, there is a continuous character χ_t of bR^n such $\chi_t \circ \alpha = e_t$. Let P be the normalized Haar measure on bR^n and $L^2(bR^n)$ be the corresponding Hilbert space based on P. One verifies easily that the representation σ of A_n in $L^2(bR^n)$ uniquely determined by

$$\sigma(u(t))F = T_{\alpha(t)}F,$$

$$(t \in R^n,\ F \in L^2(bR^n))$$

$$\sigma(v(t))F = \chi_t F$$

is semiregular and irreducible.

The Hilbert space in which σ acts is not separable ($\{\chi_t : t \in R^n\}$ is a complete orthonormal set in $L^2(bR^n)$). There exist semiregular irreducible reprensations of A_n acting in separable Hilbert spaces. For example, if γ is a non-measurable character of R^n, then the representation η of A_n determined by

$$\eta(u(t)) = \pi(u(t)),$$

$$(t \in R^n)$$

$$\eta(v(t)) = \gamma(t)\pi(v(t))$$

is semiregular, irreducible, and acts in $L^2(R^n)$.
However, if a semiregular irreducible representation of A_n is weakly measurable (as it is in the case of σ), then it must necessarily act in a non-separable Hilbert space (cf.[12]). Weak measurability must be assumed if one wants to use integration theory, thereby the appearance of non-separable Hilbert spaces is inevitable in the context of generalized Schrödinger operators.

In a coming section we shall encounter weakly measurable semiregular irreducible representations of A_n other then σ. We conclude this section by exhibiting an irregular irreducible representation of A_n. An example of such a representation is provided by τ acting in $L^2(bR^n)$ by the rule

$$\tau(u(t))F = e^{i\frac{|t|^2}{2}}\chi_t T_{\alpha(t)}F,$$

$$(F \in L^2(bR^n), t \in R^n;\ |t|^2 = t_1^2 + \ldots + t_n^2)$$

$$\tau(v(t))F = \chi_t F.$$

3. *The Burnat-Shubin-Herczyński theorem*

Let V be a real almost periodic function on R^n. Suppose $((a_\lambda^{(k)})_{\lambda\in R^n})_{k\in N}$ is a sequence in C^{R^n} such that $\overline{a_\lambda^{(k)}} = a_{-\lambda}^{(k)}$ for each k and each λ, $a_\lambda^{(k)} \neq 0$ only for finitely many λ when k is fixed, and

$$\lim_{k\to\infty} ||V - \sum_{\lambda\in R^n} a_\lambda^{(k)} e_\lambda||_\infty = 0.$$

Then the limit

$$\lim_{k\to\infty} \sum_{\lambda\in R^n} a_\lambda^{(k)} v_1(\lambda_1)\cdot\ldots\cdot v_n(\lambda_n)$$

exists and defines an element $\mathcal{V}$ in A_n. If ρ is a semiregular, irreducible representation of A_n, then $V_\rho = \rho(\mathcal{V})$ and one can define a corresponding Hamiltonian H_ρ.

The following striking result is due to Burnat, Shubin, and Herczyński (cf.[2],[6],[7],[13]).

Theorem The spectrum of H_σ is the same as the spectrum of H_π.

The recent proof of this theorem given by Krupa and Zawisza [10] uses ultrapowers of Hilbert spaces and operators. We shall briefly sketch this proof.

An ultrafilter $\mathcal{U}$ on N is a collection of subsets of N such that

(1) $A \in \mathcal{U}$ and $A \subset B \subset N$ implies $B \in \mathcal{U}$.

(2) $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$,

(3) $\emptyset \notin \mathcal{U}$,

(4) for each $A \subset N$, either $A \in \mathcal{U}$ or $N \setminus A \in \mathcal{U}$.

An ultrafilter is free if it contains the complements of all finite subsets of N. Given a bounded sequence $(a_n)_{n\in N}$ of complex numbers and a free ultrafilter $\mathcal{U}$ on N, there is $a \in C$ such that

$$\{n \in N : |a - a_n| < \varepsilon\} \in \mathcal{U}$$

for each $\varepsilon > 0$. a is called the limit of $(a_n)_{n\in N}$ with respect to $\mathcal{U}$ and is denoted by $\lim_{\mathcal{U}} a_n$. If H is a Hilbert space and $l^\infty(H)$ is the Banach space of bounded sequences in H, then

$$\{(x_n)_{n\in N} \in l^{\infty}(H) : \lim_{\mathcal{U}} ||x_n|| = 0\}$$

is a closed subspace of $l^{\infty}(H)$ and the quotient Banach space

$$H^N/\mathcal{U} = l^{\infty}(H)/\{(x_n)_{n\in N} \in l^{\infty}(H) : \lim_{\mathcal{U}} ||x_n|| = 0\}$$

can be converted into a Hilbert space by taking

$$([(x_n)_{n\in N}], [(y_n)_{n\in N}]) = \lim_{\mathcal{U}}(x_n, y_n)$$

as scalar product. $H^N/\mathcal{U}$ is called the ultrapower of H with respect to $\mathcal{U}$. For any bounded operator T in H, $T^N/\mathcal{U}$ defined as

$$(T^N/\mathcal{U})[(x_n)_{n\in N}] = [(Tx_n)_{n\in N}]$$

is a bounded operator in $H^N/\mathcal{U}$; it is called the ultrapower of T with respect to $\mathcal{U}$. If λ is in the resolvent set of T and $R_\lambda(T)$ is the resolvent of T at λ, then

$$R_\lambda(T)(\lambda I-T) = (\lambda I-T)R_\lambda(T) = I \qquad (I \text{ the identity operator})$$

and passing to ultrapowers gives

$$(R_\lambda(T)^N/\mathcal{U})(\lambda I-T^N/\mathcal{U}) = (\lambda I-T^N/\mathcal{U})(R_\lambda(T)N/\mathcal{U}) = I,$$

which shows that λ is also in the resolvent set of $T^N/\mathcal{U}$ and $R_\lambda(T^N/\mathcal{U}) = R_\lambda(T)^N/\mathcal{U}$. Hence, letting $\sigma(A)$ denote the spectrum of the operator A, we have $\sigma(T^N/\mathcal{U}) \subset \sigma(T)$. If E is a reducing subspace for $T^N/\mathcal{U}$, then $\sigma((T^N/\mathcal{U})|E) \subset \sigma(T^N/\mathcal{U})$ and so $\sigma((T^N/\mathcal{U})|E) \subset \sigma(T)$.

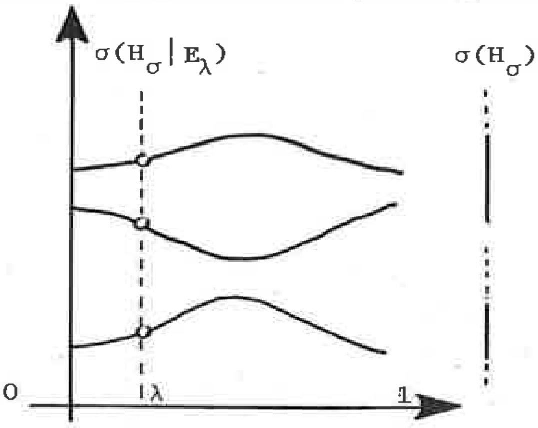
It can be shown (this is the main technical ingredient of the proof) that there exist two isometries $j:L^2(bR^n) \to L^2(R^m)^N/\mathcal{U}$ and $k:L^2(R^n) \to B^2(R^n)^N/\mathcal{U}$ such that, for each $\lambda \in C\backslash R$, $j(L^2(bR^n))$ is a reducing subspace for $R_\lambda(H_\sigma)^N/\mathcal{U}$, and

$$jR_\lambda(H_\sigma)j^{-1} = R_\lambda(H_\pi)^N/\mathcal{U} \quad \text{on} \quad j(L^2(bR^n)),$$

$$kR_\lambda(H_\pi)k^{-1} = R_\lambda(H_\sigma)^N/\mathcal{U} \quad \text{on} \quad k(L^2(R^n)).$$

First of these equalities guarantees that $\sigma(R_\lambda(H_\sigma)) \subset \sigma(R_\lambda(H_\pi))$, the other implies that $\sigma(R_\lambda(H_\pi)) \subset \sigma(R_\lambda(H_\sigma))$. Thus $\sigma(R_\lambda(H_\sigma)) = \sigma(R_\lambda(H_\pi))$ and finally, by the spectral mapping theorem, $\sigma(H_\sigma) = \sigma(H_\pi)$.

By way of application, let us consider the classical problem of qualitative determination of the spectrum of a one-dimensional Schrödinger operator with a periodic potential with period 2π. For each $\lambda \in [0,1)$, let E_λ be the closed subspace of $L^2(b\mathbb{R})$ spanned by $\{\chi_{\lambda+n}: n \in Z\}$. It can be shown without much difficulty that each E_λ is an invariant subspace for H_σ and that $H_\sigma|E_\lambda$ has compact resolvent. Thus each $\sigma(H_\sigma|E_\lambda)$ is a discrete subset of R unbounded from above. As λ varies continuously over $[0,1)$, $\sigma(H_\sigma|E_\lambda)$ also varies continuously, and since the $\sigma(H_\sigma|E_\lambda)$ sum up to yield $\sigma(H_\sigma)$, the latter set has a characteristic band structure. By the Burnat-Shubin-Herczyński theorem, the same is true of $\sigma(H_\pi)$.



It is worth noting that while the sets $\sigma(H_\pi)$ and $\sigma(H_\sigma)$ coincide, the types of the spectra of H_π and H_σ are different. Indeed, a careful analysis shows that H_π has purely absolutely continuous spectrum, whereas H_σ has pure point spectrum.

As another consequence of the Burnat-Shubin-Herczyński theorem, we mention the spectral mixing theorem (cf.[7]). It states that for any real almost periodic functions $V_i (1 \le i \le K)$ on R^n and any mollified characteristic functions $\rho_i (1 \le i \le K)$ of disjoint cones in R^n, if we let $H_\pi^{(i)}$ be the Schrödinger operators with potentials V_i and H_π^{mix} be the Schrödinger operator with potential $\sum_{i=1}^{K} \rho_i V_i$, then

$$\bigcup_{i=1}^{K} \sigma(H_\pi^{(i)}) \subset \sigma(H_\pi^{mix}).$$

4. *First order analogues of* H_σ

The spectral analysis of H_σ for a general almost periodic potential is a challenging problem. One can get some idea of the complexity of such analysis by examining first order analogues of H_σ.

Let V be a real almost periodic function on R. Let $P^{(\pi)}$ and $P^{(\sigma)}$ be the momentum operators associated with the representations π and σ, respectively, of A_1. Let

$$A_\pi = P^{(\pi)} + V_\pi$$

and

$$A_\sigma = P^{(\sigma)} + V_\sigma.$$

Let U be the unitary operator in $L^2(R)$ given by

$$Uf = uf \qquad (f \in L^2(R)),$$

where

$$u(x) = \exp(-i \int_0^x V(t)dt) \qquad (x \in R).$$

Then $A_\pi = UP_\pi U^{-1}$. In particular, A_π has purely absolutely continuous spectrum. A similar statement for A_σ is false. It turns out that the spectrum of A_σ is either pure point, or purely singular continuous, or purely absolutely continuous, and that each of these cases can occur (cf.[3]). Determination of the type of the spectrum of A_σ is related to some cohomology theory which proved to be important in harmonic analysis and ergodic theory (cf.[5]). By using the Trotter product formula, one can show that the unitary group $(U_t)_{t\in R}$ generated by iA_σ takes the form

$$U_tF = Y_t T_{\alpha(t)} F \qquad (t \in R,\ F \in L^2(bR),$$

where $Y: (t, \omega) \to Y_t(\omega)$ is a cocycle on bR, i.e., a continuous function on $R \times bR$ with values of modulus 1 such that, for all $s, t \in R$ and all $\omega \in bR$,

$$Y_{s+t}(\omega) = Y_s(\omega)Y_t(\omega+\alpha(s)).$$

Y is said to be a coboundary if there exists an invariant section of Y, i.e., a measurable function X on bR with values of modulus 1 such that, given $t \in R$,

$$Y_t(\omega) = X(\omega)\overline{X(\omega+\alpha(t))}$$

for $\mathbb{P}$-almost all ω in bR. It can be shown that A_σ has pure point spectrum if and only if Y is a coboundary.

The complicated character of the spectrum of A_σ is reflected in an unexpected manner in the harmonic-analytic properties of the function u defined above, which is a generalized eigenfunction of A_π. It can be shown, among other things, that if A_σ has purely continuous spectrum, then, for each $\lambda \in \mathbb{R}$

$$Fu(\lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(x)e^{-i\lambda x}dx = 0$$

(cf.[3],[4]). From the standpoint of the classical harmonic analysis this result is quite astonishing, for the usual constructions of functions f in $L^\infty(\mathbb{R})$ such that $Ff(\lambda) = 0$ for each $\lambda \in \mathbb{R}$

$$\lim_{T\to\infty} \inf \frac{1}{2T} \int_{-T}^{T} |f(x)|^2 dx > 0$$

are very sophisticated (cf.[8];[9], Chap. 6).

We recall that a linear continuous functional m on $L^\infty(\mathbb{R})$ is said to be a Banach mean on $L^\infty(\mathbb{R})$ if it is satisfies the following conditions:

(1) $m(1) = 1 = ||m||$,

(2) $m(T_x f) = m(f)$ for $f \in L^\infty(\mathbb{R})$ and $x \in \mathbb{R}$.

As is known, the set of all Banach means on $L^\infty(\mathbb{R})$ has at least the cardinality of the hypercontinuum (cf. [11]). Given a Banach mean m on $L^\infty(\mathbb{R})$, a function f in $L^\infty(\mathbb{R})$, we let $F_m f$ denote the Fourier transform of f with respect to m, defined as

$$F_m f(\lambda) = m(fe_{-\lambda}) \qquad (\lambda \in \mathbb{R}).$$

One can show that if Y is a coboundary whose no invariant section is continuous, then the set

$$\{(F_m u(\lambda))_{\lambda\in\mathbb{R}} : m \text{ a Banach mean on } L^\infty(\mathbb{R})\}$$

is $\mathbb{C}$-linearly isomorphic to a closed convex set in $\mathbb{C}$ different from a singleton (cf.[3],[4]). The existence of a function in $L^\infty(\mathbb{R})$ with the Fourier transforms with respect to Banach means as above is a very strange phenomenon, and one wonders whether this phenomenon can easily be exhibited without making appeal to A_σ.

We close this section with a remark on weakly measurable semiregular irreducible representations of A_n. With A_σ there is associated a weakly measurable semiregular irreducible representation κ of A_1 in $L^2(b\mathbb{R})$

determined by

$$\kappa(u(t)) = U_t F,$$
$$(t \in R,\ F \in L^2(b\mathbb{R}))$$
$$\kappa(v(t)) = \chi_t F$$

(recall that $(U_t)_{t\in R}$ is the unitary group generated by iA_σ). Operators A_σ with different types of spectrum lead to inequivalent representations of A_1. Now it is clear that some inequivalent weakly measurable semiregular irreducible representations of A_n can be constructed by taking the tensor products of semiregular irreducible representations of A_1 of the above form and, possibly, of copies of the regular irreducible representation of A_1.

5. *Conclusion*

It was our intention in this article to present a circle of ideas related to the spectral analysis of non-standard Hamiltonians. Some problems in this area alluded to above have been solved, but many more have not yet been seriously attacked. Among the latter there are some which we find to be of particular interest for further study:

(1) Determine all, up to equivalence, weakly measurable semiregular irreducible representations of A_n.

(2) Does the Burnat-Shubin-Herczyński theorem remain true if σ is replaced by any weakly measurable semiregular irreducible representation of A_n?

(3) What is the general relation between non-standard Hamiltonians and the generalized eigenfunctions of usual Hamiltonians?

REFERENCES

[1] BRATELLI O, ROBINSON D.W. Operator Algebras and Quantum Statistical Mechanics, Vol. I and II, Springer, New York 1979 and 1981.

[2] BURNAT M. "Die Spektraldarstellung einiger Differentialoperatoren mit periodischen Koeffizienten im Raume der Fastperiodischen Funktionen", Studia Math., 25 (1964), 33-64.

[3] CHOJNACKI W. "The generalized spectral analysis of $i^{-1}\frac{d}{dx} + q$ with

real almost periodic q, with applications to harmonic analysis", Rend. Math., (7) 4 (1984), 221-240.

[4] CHOJNACKI W. "Cocycles and almost periodicity", J. London Math. Soc., (2) 35 (1987), 98-108.

[5] HELSON H. "Analyticity on compact Abelian groups", in Algebras in Analysis (J.H. Williamson, Ed.), pp. 1-62, Academic Press, London, 1975.

[6] HERCZYŃSKI J. "A note on Shubin's theorem", Bull. Acac. Polon. Sci. Sér. Sci. Math., 29 (1981), 65-71.

[7] HERCZYŃSKI J. "Schrödinger operators with almost periodic potentials in non-separable Hilbert spaces", Banach Center Publ., in press.

[8] KAHANE J.-P. "Sur les fonctions presque périodiques généralisées dont le spectre est vide", Studia Math., 21 (1962) 231-236.

[9] KAHANE J.-P. Some Random Series of Functions, Cambridge University Press, Cambridge 1985.

[10] KRUPA A., ZAWISZA B. "Ultrapowers of unbounded selfadjoint operators", Studia Math., in press.

[11] ROSENBLATT J.M. "The number of extensions of an invariant mean", Compositio Math., 33 (1976), 147-159.

[12] SEGAL I.E., VON NEUMAN J. "A theorem on unitary representations of semisimple Lie groups", Ann. of Math., (2) 52 (1950), 509-517.

[13] SHUBIN M.A. "Theorems on the coincidence of the spectra of a pseudo-differential almost periodic operator in the space $L^2(R^n)$ and $B^2(R^n)$", Sibirsk. Mat. Zh., 17 (1976).

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