A Statistical Rationalisation of Hartley's Normalised Eight-Point Algorithm

Wojciech Chojnacki, Michael J. Brooks,
Anton van den Hengel, Darren Gawley

School of Computer Science, University of Adelaide,
Adelaide, SA 5005, Australia

`{wojtek,mjb,hengel,dg}@cs.adelaide.edu.au`

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Key points

- **Hartley's normalised eight-point algorithm can be justified statistically**
- **validation procedure**
  - **identify a cost function**
  - **link the cost function with a statistical model**

Related work

- **Longuet-Higgins (Nature, 1981)**
- **Hartley (PAMI, 1997)**
- **Torr (PhD, 1995)**
- **Mühlich and Mester (ECCV, 1998)**
- **Torr and Fitzgibbon (BMVC, PAMI, 2003)**

Set-up

- $\boldsymbol{m} = [m_1, m_2, 1]^T$ **— image point**
- $(\boldsymbol{m}, \boldsymbol{m}')$ **— pair of corresponding points**
- $\boldsymbol{F} = [f_{ij}]$ **—** 3×3 ***fundamental matrix*****; captures**
  - **relative orientation of the cameras**
  - **internal geometry of the cameras**
- ***epipolar constraint***

$$\boldsymbol{m}'^T \boldsymbol{F} \boldsymbol{m} = 0$$

- ***singularity constraint***

$$\det \boldsymbol{F} = 0$$

Algebraic Least Squares

- $\boldsymbol{x} = [m_1, m_2, m'_1, m'_2]^T$ **— compact descriptor of** $(\boldsymbol{m}, \boldsymbol{m}')$
- $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ **— data points**
- ***algebraic least squares* cost function**

$$J_{\mathrm{ALS}}(\boldsymbol{F}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \frac{\sum_{i=1}^{n} (\boldsymbol{m}_i'^T \boldsymbol{F} \boldsymbol{m}_i)^2}{\|\boldsymbol{F}\|_F^2}$$

- $\|\boldsymbol{F}\|_F = (\sum_{i,j} f_{ij}^2)^{1/2}$ **— the Frobenius norm**
- ***algebraic least squares* estimate**

$$\widehat{\boldsymbol{F}}_{\mathrm{ALS}} = \arg\min_{\boldsymbol{F} \neq \boldsymbol{0}} J_{\mathrm{ALS}}(\boldsymbol{F}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$$

ALS Reformulated

- **matrix algebra**

$$\boldsymbol{\theta} = \mathrm{vec}(\boldsymbol{F}^T) \qquad \boldsymbol{u}(\boldsymbol{x}) = \mathrm{vec}(\boldsymbol{m}\boldsymbol{m}'^T)$$

$$\boldsymbol{m}'^T\boldsymbol{F}\boldsymbol{m} = \boldsymbol{\theta}^T\boldsymbol{u}(\boldsymbol{x}) \qquad \boldsymbol{A} = \sum_{i=1}^{n} \boldsymbol{u}(\boldsymbol{x}_i)\boldsymbol{u}(\boldsymbol{x}_i)^T$$

$$J_{\mathrm{ALS}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \frac{\boldsymbol{\theta}^T\boldsymbol{A}\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|^2} \qquad \|\boldsymbol{\theta}\| = (\theta_1^2 + \cdots + \theta_9^2)^{1/2}$$

- $\widehat{\theta}_{\mathrm{ALS}}$ **is an eigenvector of** A **associated with the smallest eigenvalue**

Data Normalisation

- **centroids**

$$\overline{\boldsymbol{m}} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{m}_i \qquad \overline{\boldsymbol{m}}' = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{m}'_i$$

- **scales**

$$s = \left(\frac{1}{2n}\sum_{i=1}^{n}(m_{1,i}-\overline{m}_1)^2 + (m_{2,i}-\overline{m}_2)^2\right)^{1/2}$$

$$s' = \left(\frac{1}{2n}\sum_{i=1}^{n}(m'_{1,i}-\overline{m}'_1)^2 + (m'_{2,i}-\overline{m}'_2)^2\right)^{1/2}$$

Data Normalisation (Cnt'd)

- **normalised data**

$$\tilde{\boldsymbol{m}}_i = \left[\frac{m_{1,i} - \overline{m}_1}{s}, \frac{m_{2,i} - \overline{m}_2}{s}, 1\right]^T$$

$$\tilde{\boldsymbol{m}}'_i = \left[\frac{m'_{1,i} - \overline{m}'_1}{s'}, \frac{m'_{2,i} - \overline{m}'_2}{s'}, 1\right]^T$$

- **equivalent form**

$$\tilde{\boldsymbol{m}}_i = \boldsymbol{T}\boldsymbol{m}_i, \qquad \tilde{\boldsymbol{m}}'_i = \boldsymbol{T}'\boldsymbol{m}'_i$$

$$\boldsymbol{T} = \begin{bmatrix} s^{-1} & 0 & -s^{-1}\overline{m}_1 \\ 0 & s^{-1} & -s^{-1}\overline{m}_2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{T}' = \begin{bmatrix} s'^{-1} & 0 & -s'^{-1}\overline{m}'_1 \\ 0 & s'^{-1} & -s'^{-1}\overline{m}'_2 \\ 0 & 0 & 1 \end{bmatrix}$$

Hartley's Approach

- **conformal transformation**

$$\tilde{\boldsymbol{F}} = \boldsymbol{T}'^{-T} \boldsymbol{F} \boldsymbol{T}^{-1} \qquad \boldsymbol{m}'^{T} \boldsymbol{F} \boldsymbol{m} = \tilde{\boldsymbol{m}}'^{T} \tilde{\boldsymbol{F}} \tilde{\boldsymbol{m}}$$

- ***Hartley* estimate**

$$\tilde{\boldsymbol{x}}_i = [\tilde{m}_{1,i}, \tilde{m}_{2,i}, \tilde{m}'_{1,i}, \tilde{m}'_{2,i}]^T$$

$$\widehat{\tilde{\boldsymbol{F}}}_{\mathrm{ALS}} = \arg\min_{\boldsymbol{F} \neq \boldsymbol{0}} J_{\mathrm{ALS}}(\boldsymbol{F}; \tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_n)$$

$$\widehat{\boldsymbol{F}}_{\mathrm{HRT}} = \boldsymbol{T}'^{T} \widehat{\tilde{\boldsymbol{F}}}_{\mathrm{ALS}} \boldsymbol{T}$$

- **as an eigenvector, $\widehat{\tilde{\boldsymbol{F}}}_{\mathrm{ALS}}$ is less "wobbly" than $\widehat{\boldsymbol{F}}_{\mathrm{ALS}}$**

Normalised Algebraic Least Squares

- ***normalised*** **algebraic least squares cost function**

$$J_{\mathrm{NALS}}(\boldsymbol{F}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \frac{\sum_{i=1}^{n} (\boldsymbol{m}_i'^T \boldsymbol{F} \boldsymbol{m}_i)^2}{\|\boldsymbol{T}'^{-T} \boldsymbol{F} \boldsymbol{T}^{-1}\|_F^2}$$

- **key property**

$$J_{\mathrm{NALS}}(\boldsymbol{F}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = J_{\mathrm{ALS}}(\tilde{\boldsymbol{F}}; \tilde{\boldsymbol{x}}_1, \ldots, \tilde{\boldsymbol{x}}_n)$$

- **fundamental consequence**

$$\widehat{\boldsymbol{F}}_{\mathrm{HRT}} = \widehat{\boldsymbol{F}}_{\mathrm{NALS}}$$

Alternative Form of J_{NALS}

- **auxiliary** 9×9 **matrix**

$$\boldsymbol{C}(s, t, \boldsymbol{m}, \boldsymbol{n}) = (t^2 \boldsymbol{I}^* + \boldsymbol{n}\boldsymbol{n}^T) \otimes (s^2 \boldsymbol{I}^* + \boldsymbol{m}\boldsymbol{m}^T)$$

$$\boldsymbol{I}^* = \mathrm{diag}(1, 1, 0)$$

$\otimes$ **denotes *Kronecker product***

- **rewriting** J_{NALS}

$$J_{\mathrm{NALS}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \frac{\boldsymbol{\theta}^T \boldsymbol{A} \boldsymbol{\theta}}{\boldsymbol{\theta}^T \boldsymbol{C} \boldsymbol{\theta}} \qquad \boldsymbol{C} = \boldsymbol{C}(s, s', \overline{\boldsymbol{m}}, \overline{\boldsymbol{m}}')$$

- $\widehat{\boldsymbol{\theta}}_{\mathrm{NALS}}$ **is a solution of the *generalised* eigenvalue problem** $\boldsymbol{A}\boldsymbol{\theta} = \lambda \boldsymbol{C}\boldsymbol{\theta}$ **corresponding to the smallest eigenvalue**

Towards a Statistical Model

- $r_i = \boldsymbol{m}_i'^T \boldsymbol{F} \boldsymbol{m}_i$ — ith ***residual*** **with** $\boldsymbol{F}$ **normalised**
- **different residuals carry different statistical weight**
- **replace** $\sum_{i=1}^{n} r_i^2$ **by** $\sum_{i=1}^{n} r_i^2 / \mathrm{var}\left[r_i\right]$
- **goal: a statistical model for data distribution under which**

$$\sum_{i=1}^{n} \frac{r_i^2}{\mathrm{var}\left[r_i\right]} = J_{\mathrm{NALS}}$$

Statistical Model

$$\mathsf{m}_i = \overline{\mathsf{m}} + \Delta\mathsf{m}_i \qquad \mathsf{m}'_i = \overline{\mathsf{m}}' + \Delta\mathsf{m}'_i$$

- $\overline{\mathsf{m}}$**,** $\overline{\mathsf{m}}'$ **— fixed, non-random locations**
- $\Delta\mathsf{m}_i$**,** $\Delta\mathsf{m}'_i$ **— random perturbations**
- $\overline{\mathsf{m}}$ **and** $\overline{\mathsf{m}}'$ **— 'true' locations bound by** $\boldsymbol{F}$

$$\overline{\mathsf{m}}'^T \boldsymbol{F} \overline{\mathsf{m}} = \boldsymbol{\theta}^T \operatorname{vec}(\overline{\mathsf{m}}\,\overline{\mathsf{m}}'^T) = 0$$

- $\Delta\mathsf{m}_i$**,** $\Delta\mathsf{m}'_j$ **are independent**
- $\mathsf{E}\left[\Delta\mathsf{m}_i\right] = \mathsf{E}\left[\Delta\mathsf{m}'_i\right] = \mathbf{0}$
- **there exist** $\sigma > 0$ **and** $\sigma' > 0$ **such that**

$$\mathsf{E}\left[\Delta\mathsf{m}_i(\Delta\mathsf{m}_i)^T\right] = \sigma^2 \boldsymbol{I}^* \qquad \mathsf{E}\left[\Delta\mathsf{m}'_i(\Delta\mathsf{m}'_i)^T\right] = \sigma'^2 \boldsymbol{I}^*$$

Statistical Model: Consequences

- **the stochastic residuals**

$$\mathsf{r}_i = \mathsf{m}_i'^T \boldsymbol{F} \mathsf{m}_i$$

have common variance

$$v = \boldsymbol{\theta}^T \boldsymbol{C}(\sigma, \sigma', \overline{\mathsf{m}}, \overline{\mathsf{m}}') \boldsymbol{\theta}$$

- **the random template for a cost function reduces to**

$$\sum_{i=1}^{n} \frac{\mathsf{r}_i^2}{\mathrm{var}\left[\mathsf{r}_i\right]} = \frac{\sum_{i=1}^{n} \mathsf{r}_i^2}{v}$$

From *Random* to *Deterministic*

- **replace** r_i **by** r_i
- **treat** $\overline{\boldsymbol{m}}$**,** $\overline{\boldsymbol{m}}'$**,** s**,** s' **as *estimates* of** $\overline{\mathsf{m}}$**,** $\overline{\mathsf{m}}'$**,** σ**,** σ'
- **replace** $\boldsymbol{C}(\sigma, \sigma', \overline{\mathsf{m}}, \overline{\mathsf{m}}')$ **by** $\boldsymbol{C} = \boldsymbol{C}(s, s', \overline{\boldsymbol{m}}, \overline{\boldsymbol{m}}')$
- **the upshot**

$$\frac{\sum_{i=1}^{n} \mathsf{r}_i^2}{v} \qquad \Longrightarrow \qquad \frac{\sum_{i=1}^{n} r_i^2}{\boldsymbol{\theta}^T \boldsymbol{C} \boldsymbol{\theta}} = J_{\mathrm{NALS}}$$

Conclusion

- Hartley's famous normalised eight-point algorithm
  - can be viewed as a means of finding a minimiser of a specific cost function
  - has a novel statistical interpretation
  - can be placed more clearly within the spectrum of methods available