

# ON THE FAVARD CLASSES OF SEMIGROUPS ASSOCIATED WITH PSEUDO-RESOLVENTS

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ABSTRACT. A pseudo-resolvent on a Banach space, indexed by positive numbers and tempered at infinity, gives rise to a bounded strongly continuous one-parameter semigroup  $S$  on a closed subspace of the ambient Banach space. We prove that the range space of the pseudo-resolvent contains the domain of the generator of  $S$ , and is contained in the Favard class of  $S$ , which consists of all uniformly Lipschitz vectors for  $S$ . We explore when some or all of these three spaces coincide.

## 1. INTRODUCTION

Let  $\mathbf{A}$  be a Banach algebra over the field  $\mathbb{C}$  of complex numbers. A family  $\{r_\lambda\}_{\lambda \in U}$  of elements of  $\mathbf{A}$ , indexed by a subset  $U \subset \mathbb{C}$ , is called a *pseudo-resolvent* in  $\mathbf{A}$  if the following *Hilbert equation* is satisfied:

$$(1) \quad r_\lambda - r_\mu = (\mu - \lambda)r_\lambda r_\mu \quad (\lambda, \mu \in U).$$

Denote by  $\mathbb{R}$  the set of all real numbers, by  $\mathbb{R}_+$  the set of all nonnegative numbers, and by  $\mathbb{R}_+^\bullet$  the set of all positive numbers. Given a pseudo-resolvent  $r = \{r_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  in  $\mathbf{A}$ , let

$$c_r = \sup\{\lambda^n \|r_\lambda^n\| \mid n \in \mathbb{N}, \lambda \in \mathbb{R}_+^\bullet\}.$$

A pseudo-resolvent  $r$  indexed by  $\mathbb{R}_+^\bullet$  and such that  $c_r$  is finite will be said to be *tempered at infinity*. Note, immediately, that if  $r$  is tempered at infinity, then  $\lim_{\mu \rightarrow \infty} r_\mu = 0$ , and so, in view of (1),

$$(2) \quad r_\lambda = \lim_{\mu \rightarrow \infty} \mu r_\mu r_\lambda$$

for each  $\lambda \in \mathbb{R}_+^\bullet$ .

Hereafter we assume that all vector spaces and algebras considered are complex.

Let  $E$  be a Banach space, and let  $\mathcal{L}(E)$  be the Banach algebra of all continuous linear operators in  $E$  equipped with the usual norm derived from that of  $E$ . Let  $A$  be a closed linear operator on  $E$  with domain  $\mathcal{D}_A$  and range  $\mathcal{R}_A$ . The *resolvent set*  $\rho_A$  of  $A$  is the set of  $\lambda \in \mathbb{C}$  for which  $\lambda - A$  is one-to-one with range equal to  $E$ . For each  $\lambda \in \rho_A$ , setting

$$R_\lambda(A) = (\lambda - A)^{-1}$$

defines the *resolvent operator* of  $A$  corresponding to  $\lambda$ ; it is bounded by the closed graph theorem. The resolvent set  $\rho_A$  is open, and the family  $R(A) = \{R_\lambda(A)\}_{\lambda \in \rho_A}$  is a special instance of pseudo-resolvent, called the *resolvent* of  $A$ .

Let  $R = \{R_\lambda\}_{\lambda \in U}$  with  $U \subset \mathbb{C}$  be a pseudo-resolvent in  $\mathcal{L}(E)$ . It immediately follows from the Hilbert equation (1) that all the  $R_\lambda$  have a common null space and a common range. These spaces are called the null space and range space of  $R$ , and are denoted  $\mathcal{N}_R$

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and  $\mathcal{R}_R$ , respectively. If  $\mathcal{N}_R$  is zero, then  $R$  is a resolvent of a closed operator whose domain coincides with  $\mathcal{R}_R$ .

Let  $S = \{S_t\}_{t \in \mathbb{R}_+}$  be a bounded strongly continuous one-parameter semigroup of operators in  $\mathcal{L}(E)$ . The boundedness of  $S$  means, of course, that

$$m_S := \sup_{t \in \mathbb{R}_+} \|S_t\| < +\infty,$$

and strong continuity is synonymous with the continuity in the strong operator topology on  $\mathcal{L}(E)$ . Let  $A$  be the (infinitesimal) generator of  $S$ . The operator  $A$  is closed, has dense domain, and its resolvent set contains  $\mathbb{R}_+^\bullet$ . The resolvent  $R(A) = \{R_\lambda(A)\}_{\lambda \in \mathbb{R}_+^\bullet}$  is tempered at infinity and  $c_{R(A)} = m_S$ . For each  $\lambda \in \mathbb{R}_+^\bullet$ ,  $R_\lambda(A)$  can explicitly be expressed in terms of  $S$  as follows:

$$(3) \quad R_\lambda(A)x = \int_{\mathbb{R}_+} e^{-\lambda t} S_t x \, dt \quad (x \in E).$$

The Favard class of  $S$ ,  $\text{Fav}(S)$ , is the subspace of  $E$  defined by

$$\text{Fav}(S) = \{x \in E \mid \limsup_{t \rightarrow +0} t^{-1} \|S_t x - x\| < +\infty\}$$

(see [3, p. 84]). In this definition, the upper limit can safely be replaced by the lower limit (cf. [3, proof of Theorem 2.1.2(a)]). As a consequence of  $S$  being bounded, it is easily seen that  $\text{Fav}(S)$  coincides with the space of all *uniformly Lipschitz vectors* for  $S$ ; that is

$$\text{Fav}(S) = \{x \in E \mid \sup_{t \in \mathbb{R}_+^\bullet} t^{-1} \|S_t x - x\| < +\infty\}$$

(see [5, Def. 3.17]). A basic, simple result is that

$$(4) \quad \mathcal{D}_A \subset \text{Fav}(S)$$

(cf. [3, Thm. 2.1.2], [5, p. 67]).

Suppose now that  $R = \{R_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$ , a pseudo-resolvent in  $\mathcal{L}(E)$ , is tempered at infinity. It is an immediate consequence of (1) that, for every  $\lambda \in \mathbb{R}_+^\bullet$ ,  $\mathcal{R}_R$  is invariant for  $R_\lambda$ . Clearly, the closure  $\overline{\mathcal{R}_R}$  of  $\mathcal{R}_R$  is also invariant for all the  $R_\lambda$ . For each  $\lambda \in \mathbb{R}_+^\bullet$ , define

$$\mathbf{r}(R)_\lambda = R_\lambda \upharpoonright \overline{\mathcal{R}_R},$$

where the symbol  $S \upharpoonright D$  denotes the restriction of the operator  $S$  to  $D$ . Obviously,  $\mathbf{r}(R) = \{\mathbf{r}(R)_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  is a pseudo-resolvent in  $\mathcal{L}(\overline{\mathcal{R}_R})$ . Observe that, on account of (2),

$$x = \lim_{\mu \rightarrow \infty} \mu R_\mu x$$

for each  $x \in \mathcal{R}_R$ . As  $\{\mu \|R_\mu\|\}_{\mu \in \mathbb{R}_+^\bullet}$  is bounded, this equality extends over all  $x \in \overline{\mathcal{R}_R}$ . Hence we immediately find that  $\mathcal{N}_{\mathbf{r}(R)}$  is zero and that  $\mathcal{R}_{\mathbf{r}(R)}$  is dense in  $\overline{\mathcal{R}_R}$ . Consequently,  $\mathbf{r}(R)$  is a resolvent of a closed densely defined operator  $A$  on  $\overline{\mathcal{R}_R}$ . Of course,  $\mathcal{D}_A = \mathcal{R}_{\mathbf{r}(R)}$ . Taking into account that  $c_{\mathbf{r}(R)} \leq c_R$  and applying the Hille–Yosida Theorem (cf. [3, Thm. 1.3.6], [8, Thm. 2.21], [12, p. 358]), we see that  $A$  is the generator of a bounded strongly continuous semigroup  $S = \{S_t\}_{t \in \mathbb{R}_+}$  on  $\overline{\mathcal{R}_R}$ , satisfying  $m_S = c_{\mathbf{r}(R)}$ . This semigroup will be termed the *semigroup associated with  $R$* . Semigroups associated with pseudo-resolvents tempered at infinity arise naturally in Markov processes theory (cf. [9, Chap. XII, §5, p. 290–302], [15]).

For a semigroup  $S$  associated with a pseudo-resolvent  $R$ , inclusion (4) can be restated as

$$\mathcal{R}_{\mathbf{r}(R)} \subset \text{Fav}(S).$$

It emerges that, in view of the special nature of  $S$ , the above relation can be refined. In this note, we prove the following:

**Theorem 1.** *Let  $E$  be a Banach space, let  $R = \{R_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  be a pseudo-resolvent in  $\mathcal{L}(E)$ , tempered at infinity, and let  $S$  be the semigroup associated with  $R$ . Then*

$$(5) \quad \mathcal{R}_{r(R)} \subset \mathcal{R}_R \subset \text{Fav}(S).$$

Observe that, of the two inclusions above, the one on the left-hand side is trivial, so only the inclusion on the right-hand side needs to be verified.

In addition, we tackle the problem of determining when some or both of the inclusions in (5) can be replaced by equalities. We prove that for dual pseudo-resolvents (on dual Banach spaces) the right-hand side inclusion reduces to equality. As a complementary result, a class of dual pseudo-resolvents is revealed for which the left-hand side inclusion is strict. It is also shown that, for pseudo-resolvents on reflexive Banach spaces, both inclusions become equalities. Finally, two examples are given to demonstrate that, in general, both inclusions in (5) can be simultaneously strict.

Interestingly, A. Bobrowski has recently given a necessary and sufficient condition for the left-hand side inclusion in (5) to become equality (cf. [2, Prop. 2.2]).

## 2. PROOF OF THE MAIN RESULT

This section is devoted to the proof of Theorem 1. We start by presenting a preliminary material.

Denote by  $\lambda_{\mathbb{R}_+}$  the Lebesgue measure on  $\mathbb{R}_+$ . As is customary, abbreviate  $d\lambda_{\mathbb{R}_+}(t)$  to  $dt$ . Let  $L^1(\mathbb{R}_+)$  be the space of equivalence classes (under equality  $\lambda_{\mathbb{R}_+}$ -almost everywhere) of complex-valued Lebesgue integrable functions  $f$  on  $\mathbb{R}_+$ . With the addition and scalar multiplication derived from the pointwise addition and scalar multiplication of the functions, and with the norm given by

$$\|f\|_1 = \int_{\mathbb{R}_+} |f(t)| dt$$

(where the same symbol  $f$  is used to denote both a function and its equivalence class),  $L^1(\mathbb{R}_+)$  is a complex Banach space. With convolution

$$(f * g)(t) = \int_0^t f(t-s)g(s) ds \quad (\lambda_{\mathbb{R}_+}\text{-a. e. } t \in \mathbb{R}_+)$$

as the product, it becomes a complex Banach algebra. For each  $\lambda \in \mathbb{R}$ , denote by  $\epsilon_\lambda$  the function

$$\epsilon_\lambda(t) = e^{\lambda t} \quad (t \in \mathbb{R}_+).$$

It is directly verified that the family  $\epsilon = \{\epsilon_{-\lambda}\}_{\lambda \in \mathbb{R}_+^\bullet}$  is a pseudo-resolvent in  $L^1(\mathbb{R}_+)$  satisfying  $c_\epsilon = 1$ . This special pseudo-resolvent enters critically the following result (cf. [13]; see also [1, 4]):

**Theorem 2.** *Let  $\mathbf{A}$  be a Banach algebra and let  $\{r_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  be a pseudo-resolvent in  $\mathbf{A}$ . Then  $r$  is tempered at infinity if and only if there exists a continuous Banach algebra homomorphism  $H: L^1(\mathbb{R}_+) \rightarrow \mathbf{A}$  such that  $H(\epsilon_{-\lambda}) = r_\lambda$  for each  $\lambda \in \mathbb{R}_+^\bullet$ . Furthermore, if a continuous homomorphism  $H: L^1(\mathbb{R}_+) \rightarrow \mathbf{A}$  satisfying  $H(\epsilon_{-\lambda}) = r_\lambda$  for each  $\lambda \in \mathbb{R}_+^\bullet$  exists, then it is unique and  $\|H\| = c_r$ .*

The above theorem is indicative of the role played by operator-valued homomorphisms of  $L^1(\mathbb{R}_+)$  in studying pseudo-resolvents tempered at infinity. This role is greatly reinforced by the *algebraic version* of the Hille–Yosida Theorem, which we present next (see [13] for a proof).

For a Banach algebra homomorphism  $H: L^1(\mathbb{R}_+) \rightarrow \mathcal{L}(E)$ , where  $E$  is a Banach space, denote by  $\mathcal{R}_H$  the subspace of  $E$  defined as

$$\mathcal{R}_H = \{x \in E \mid x = H(f)y \text{ for some } f \in L^1(\mathbb{R}_+) \text{ and } y \in E\}.$$

For each  $t \in \mathbb{R}_+$ , let  $\mathbb{T}_t^+ : L^1(\mathbb{R}_+) \rightarrow L^1(\mathbb{R}_+)$  be the *forward shift* by  $t$  given by

$$(\mathbb{T}_t^+ f)(x) = 1_{[t, +\infty)}(x)f(x-t) \quad (f \in L^1(\mathbb{R}_+), \lambda_{\mathbb{R}_+}\text{-a. e. } x \in \mathbb{R}_+),$$

where, as usual, the symbol  $1_A$  denotes the characteristic function of the set  $A$ . Clearly, the family  $\mathbb{T}^+ = \{\mathbb{T}_t^+\}_{t \in \mathbb{R}_+}$  is a strongly continuous semigroup of linear contractions on  $L^1(\mathbb{R}_+)$ .

**Theorem 3.** *Let  $E$  be a Banach space. If  $H: L^1(\mathbb{R}_+) \rightarrow \mathcal{L}(E)$  is a Banach algebra homomorphism, then there exists a unique strongly continuous semigroup  $S = \{S_t\}_{t \in \mathbb{R}_+}$  on  $\mathcal{R}_H$  such that, for each  $t \in \mathbb{R}_+$ , each  $f \in L^1(\mathbb{R}_+)$ , and each  $x \in E$ ,*

$$(6) \quad S_t H(f)x = H(\mathbb{T}_t^+ f)x.$$

The semigroup  $S$  is bounded and  $m_S = \|H\|$ .

We now draw out a link between Theorems 2 and 3. Let  $E$  be a Banach space and let  $R = \{R_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  be a pseudo-resolvent in  $\mathcal{L}(E)$ , tempered at infinity. By Theorem 2, there is a unique Banach algebra homomorphism  $H: L^1(\mathbb{R}_+) \rightarrow \mathcal{L}(E)$  such that  $R_\lambda = H(\epsilon_{-\lambda})$  for each  $\lambda \in \mathbb{R}_+^\bullet$ . It is clear that  $\mathcal{R}_R \subset \mathcal{R}_H$ . A consequence of the linear span of  $\{\epsilon_{-\lambda} \mid \lambda \in \mathbb{R}_+^\bullet\}$  being dense in  $L^1(\mathbb{R}_+)$  is the inclusion  $\mathcal{R}_H \subset \overline{\mathcal{R}_R}$ . It turns out that  $\mathcal{R}_H$  is closed, so in fact

$$(7) \quad \mathcal{R}_H = \overline{\mathcal{R}_R}.$$

The closedness of  $\mathcal{R}_H$  is an immediate consequence of two results:

- the simple observation that  $L^1(\mathbb{R}_+)$  possesses a bounded approximate identity; one such approximate identity is the family  $\{\lambda \epsilon_{-\lambda}\}_{\lambda \in \mathbb{R}_+^\bullet}$ ;
- a deep theorem found independently by E. Hewitt [11], P. C. Curtis and A. Figá-Talamanca [7], and S. L. Gulick, T. S. Liu and A. C. M. van Rooij [10], which generalises the so-called factorisation theorem of P. J. Cohen [6]; it states that  $\mathcal{R}_H$  is closed for any continuous homomorphism  $H: \mathbf{B} \rightarrow \mathcal{L}(E)$ , where  $\mathbf{B}$  is a Banach algebra with a bounded approximate identity, and  $E$  is a Banach space.

By virtue of Theorem 3, there is a unique semigroup  $S$  on  $\mathcal{R}_H$  for which (6) holds. We claim that  $S$  coincides with the semigroup associated with  $R$ . Indeed, by (7),  $S$  can be viewed as a semigroup on  $\overline{\mathcal{R}_R}$ . Observe that

$$(8) \quad \int_{\mathbb{R}_+} e^{-\lambda t} \mathbb{T}_t^+ f dt = \epsilon_{-\lambda} * f \quad (f \in L^1(\mathbb{R}_+)).$$

Now, if  $x \in \mathcal{R}_H$ , then there exist  $f \in L^1(\mathbb{R}_+)$  and  $y \in E$  such that  $x = H(f)y$ . In view of (8) and the boundedness of  $S$ , we have, for each  $\lambda \in \mathbb{R}_+^\bullet$ ,

$$(9) \quad \begin{aligned} \int_{\mathbb{R}_+} e^{-\lambda t} S_t x dt &= \int_{\mathbb{R}_+} e^{-\lambda t} H(\mathbb{T}_t^+ f)y dt = H \left( \int_{\mathbb{R}_+} e^{-\lambda t} \mathbb{T}_t^+ f dt \right) y \\ &= H(\epsilon_{-\lambda} * f)y = H(\epsilon_{-\lambda})H(f)y = R_\lambda x. \end{aligned}$$

Let  $A$  be the generator of  $S$ . Comparison of (3) and (9) shows that  $R_\lambda = R_\lambda(A)$  for each  $\lambda \in \mathbb{R}_+^\bullet$ . Since every operator with non-void resolvent set is uniquely determined by any of its resolvent operators, it is now clear that  $A$  coincides with the generator of the semigroup associated with  $R$ . This establishes the claim.

With these preliminaries in place, we are now in position to prove Theorem 1.

*Proof of Theorem 1.* As already noted, we only need to establish the right-hand side inclusion in (5). Let  $H: L^1(\mathbb{R}_+) \rightarrow \mathcal{L}(E)$  be a continuous homomorphism  $H: L^1(\mathbb{R}_+) \rightarrow \mathcal{L}(E)$  such that  $H(\epsilon_{-\lambda}) = R_\lambda$  for each  $\lambda \in \mathbb{R}_+^\bullet$ ; the existence of  $H$  is guaranteed by Theorem 2. Fix  $\lambda \in \mathbb{R}_+^\bullet$  arbitrarily. Given  $x \in \mathcal{R}_R$ , select  $y \in E$  so that  $x = R_\lambda y$ . In view of the remarks above, the associated semigroup  $S$  satisfies (6). Thus, for each  $t \in \mathbb{R}_+$ ,

$$(10) \quad S_t x = S_t H(\epsilon_{-\lambda}) y = H(\mathbb{T}_t^+ \epsilon_{-\lambda}) y.$$

One verifies at once that

$$\mathbb{T}_t^+ \epsilon_{-\lambda} = \epsilon_{-\lambda} + \lambda \epsilon_{-\lambda} * 1_{(0,t]} - 1_{(0,t]}.$$

It follows that

$$\|\mathbb{T}_t^+ \epsilon_{-\lambda} - \epsilon_{-\lambda}\| \leq (\lambda \|\epsilon_{-\lambda}\| + 1) \|1_{(0,t]}\| = 2t,$$

whence, by (10),

$$\sup_{t \in \mathbb{R}_+^\bullet} t^{-1} \|S_t x - x\| \leq 2 \|H\| \|y\|.$$

Thus  $x \in \text{Fav } S$ , which establishes the desired inclusion.  $\square$

### 3. THE CASE OF DUAL PSEUDO-RESOLVENTS

Let  $E$  be a Banach space and let  $E'$  be its dual space. For an operator  $T \in \mathcal{L}(E)$ , let  $T'$  denote its dual operator in  $\mathcal{L}(E')$ . The same notation applies when  $T$  is a densely defined operator on  $E$ . Recall that if  $T$  is a densely defined operator on  $E$ , then  $T'$  is closed but not necessarily densely defined on  $E'$ —the space  $\mathcal{D}_{T'}$  is merely weakly\* dense in  $E'$ .

Let  $R = \{R_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  be a pseudo-resolvent in  $\mathcal{L}(E)$ . The family  $R' = \{R'_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  is a pseudo-resolvent in  $\mathcal{L}(E')$  called the *dual pseudo-resolvent* of  $R$ . Since  $c_{R'} = c_R$ , we see that if  $R$  is tempered at infinity, then so too is  $R'$ . Assume henceforth that  $R$  is tempered at infinity. Denote by  $S$  the semigroup associated with  $R$ , and by  $S^+$  the semigroup associated with  $R'$ . Let  $A$  and  $A^+$  denote the corresponding generators. In this section, we elaborate on the inclusions in (5) for the Favard class of  $S^+$ .

**Theorem 4.** *Under the assumptions as above, we have  $\mathcal{R}_{R'} = \text{Fav}(S^+)$ .*

In order to prove the theorem, we need an auxiliary result.

**Proposition 1.** *Let  $F$  be a Banach space, and let  $T = \{T_t\}_{t \in \mathbb{R}_+}$  be a bounded strongly continuous semigroup on  $F$  with generator  $B$ . If  $x \in \text{Fav}(T)$ , then there exists a sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $\mathcal{D}_B$  such that  $\lim_{n \rightarrow \infty} x_n = x$  and  $\{Bx_n\}_{n \in \mathbb{N}}$  is bounded.*

This proposition is well known (cf. [5, p. 69]), but we give a short proof for the convenience of the reader.

*Proof of Proposition 1.* Let  $\{\lambda_n\}_{n \in \mathbb{N}}$  be a sequence of positive numbers such that  $\lim_{n \rightarrow \infty} \lambda_n = +\infty$ . Given  $x \in \text{Fav}(T)$ , set  $x_n = \lambda_n R_{\lambda_n}(B)x$  for each  $n \in \mathbb{N}$ . Clearly,  $\{x_n\}_{n \in \mathbb{N}}$  is sequence in  $\mathcal{D}_B$  such that  $x = \lim_{n \rightarrow \infty} x_n$ . We shall show that  $\{Bx_n\}_{n \in \mathbb{N}}$  is

bounded. Let  $C$  be a positive constant such that  $h^{-1}\|T_h x - x\| \leq C$  for all  $h \in \mathbb{R}_+^\bullet$ . In view of (3), for each  $n \in \mathbb{N}$  and each  $h \in \mathbb{R}_+^\bullet$ ,

$$h^{-1}(T_h x_n - x_n) = \int_{\mathbb{R}_+} h^{-1} \lambda_n e^{-\lambda_n t} T_t(T_h x - x) dt,$$

whence  $\|h^{-1}(T_h x_n - x_n)\| \leq C m_T$ , and finally  $\|B x_n\| \leq C m_T$ .  $\square$

We can now pass to proving Theorem 4.

*Proof of Theorem 4.* In view of Theorem 1, all that we need is to show that  $\text{Fav}(S^+) \subset \mathcal{R}_{R'}$ . Let  $x' \in \text{Fav}(S^+)$ . By Proposition 1, there exists a sequence  $\{x'_n\}_{n \in \mathbb{N}}$  in  $\mathcal{D}_{A^+}$  such that  $\sup_{n \in \mathbb{N}} \|A^+ x'_n\| < +\infty$  and  $\lim_{n \rightarrow \infty} \|x'_n - x'\| = 0$ . Fix  $\lambda \in \mathbb{R}_+^\bullet$  arbitrarily. Clearly, for each  $n \in \mathbb{N}$ ,

$$x'_n = R'_\lambda(\lambda - A^+)x'_n.$$

By the relative weak\* compactness of bounded sets in  $E'$ , there exists a (generalised) subsequence  $\{A^+ x'_{n_i}\}_{i \in I}$  and  $y' \in E'$  such that

$$\text{w}^*\text{-}\lim_{i \in I} A^+ x'_{n_i} = y'.$$

Since  $R'_\lambda$  is weakly\* continuous, it follows that

$$(11) \quad x' = R'_\lambda(\lambda x' - y').$$

Hence  $x' \in \mathcal{R}_{R'}$ , establishing the desired inclusion.  $\square$

A simple modification of the above proof leads to the following result:

**Theorem 5.** *If  $E$  is reflexive, then  $\mathcal{R}_{r(R)} = \mathcal{R}_R = \text{Fav}(S)$ .*

*Proof.* Assume that  $E$  is reflexive. We shall prove that

$$(12) \quad \mathcal{R}_{r(R')} = \mathcal{R}_{R'} = \text{Fav}(S^+).$$

This will be a sufficient step: the theorem follows immediately upon applying (12) to the Favard class of the semigroup  $S^{++}$  associated with the bidual pseudo-resolvent  $R''$ , modulo the observation that  $E'$  is reflexive (so that the suggested application of (12) makes sense) and the remark that  $R''$  and  $S^{++}$  are naturally identifiable with  $R$  and  $S$ , respectively.

By virtue of Theorem 1, the proof of (12) reduces to showing that  $\text{Fav}(S^+) \subset \mathcal{R}_{r(R')}$ . Let  $x' \in \text{Fav}(S^+)$ . Retaining the notation from the proof of the foregoing theorem, note that  $y'$  appearing in (11) belongs to the weak\* closure of  $\{A^+ x'_n\}_{n \in \mathbb{N}}$ , which, in view of the reflexivity of  $E$ , coincides with the weak closure of  $\{A^+ x'_n\}_{n \in \mathbb{N}}$ . Remembering that  $\mathcal{D}_{A^+} = \mathcal{R}_{r(R')} \subset \mathcal{R}_{R'}$  and noting that, being norm closed,  $\overline{\mathcal{R}_{R'}}$  is also weakly closed, we see that  $y' \in \overline{\mathcal{R}_{R'}}$ . In view of Theorem 4, we have  $x' \in \mathcal{R}_{R'}$ . Thus  $\lambda x' - y'$  falls into  $\overline{\mathcal{R}_{R'}}$ . Now (11) implies that  $x' \in \mathcal{R}_{r(R')}$ , which establishes the desired inclusion.  $\square$

We now examine more closely the case in which  $\overline{\mathcal{R}_R} = E$ . The associated semigroup  $S$  acts then on the whole of  $E$ , its generator  $A$  is densely defined on  $E$ , and  $R_\lambda = R_\lambda(A)$  for each  $\lambda \in \mathbb{R}_+^\bullet$ . The dual semigroup  $S' = \{S'_t\}_{t \in \mathbb{R}_+}$  on  $E'$  may fail to be strongly continuous; it is, however, weakly\* continuous in the sense that, for each  $x' \in E'$ , the map  $\mathbb{R}_+ \ni t \mapsto S'_t x' \in E'$  is continuous under the weak\* topology on  $E'$ . The operator  $A'$  is the weak\* generator of  $S'$  (see [3, Prop. 1.4.4], [8, Thm. 1.34], [12, p. 335–336]). The sets  $\rho_A$  and  $\rho_{A'}$  coincide and  $R_\lambda(A') = R_\lambda(A)'$  for each  $\lambda \in \rho_A$ . In particular,  $R'_\lambda = R_\lambda(A')$  for each  $\lambda \in \mathbb{R}_+^\bullet$ , which can emphatically be written as

$$(13) \quad R' = R(A').$$

The set

$$E^\circ = \{x' \in E' \mid \lim_{t \rightarrow 0^+} \|S'_t x' - x'\| = 0\}$$

is a norm closed subspace of  $E'$ . According to a theorem of Phillips [14] (see also [3, Props. 1.4.69(b) and 1.4.7(a)], [5, Thm. 3.14], [12, p. 344–345]),  $\mathcal{D}_{A'}$  is a norm dense subspace of  $E^\circ$ . Hence in particular  $E^\circ$  is weakly\* dense in  $E'$ . Coupled with (13) and  $\mathcal{D}_{A'} = \mathcal{R}_{R(A')}$ , the norm denseness of  $\mathcal{D}_{A'}$  implies also that

$$(14) \quad E^\circ = \overline{\mathcal{R}_{R'}}.$$

It is easily seen that the space  $E^\circ$  is invariant for all the  $S'_t$  and that setting

$$S_t^\circ = S'_t \upharpoonright E^\circ \quad (t \in \mathbb{R}_+)$$

defines a strongly continuous semigroup  $S^\circ = \{S_t^\circ\}_{t \in \mathbb{R}_+}$  on  $E^\circ$ . Let  $A^\circ$  be the generator of  $S^\circ$ . Another part of the theorem of Phillips mentioned above asserts that

$$\mathcal{D}_{A^\circ} = \{x' \in \mathcal{D}_{A'} \mid A'x' \in E^\circ\}$$

and

$$A^\circ = A' \upharpoonright \mathcal{D}_{A^\circ}.$$

(cf. [3, Prop. 1.4.7(b)]). Since  $m_{S^\circ} \leq m_{S'} = m_S$ , we have  $\mathbb{R}_+^\bullet \subset \rho_{A^\circ}$ , which together with the above characterisation of  $A^\circ$  immediately implies that, for each  $\lambda \in \mathbb{R}_+^\bullet$ ,

$$(15) \quad R_\lambda(A^\circ) = R_\lambda(A') \upharpoonright E^\circ.$$

Now, taking into account (13) and (14), we see that

$$(16) \quad r(R') = R(A^\circ).$$

It is also clear that  $A^\circ$  coincides with  $A^+$ , and that  $S^\circ$  coincides with  $S^+$ .

We are now prepared to state our next result.

**Theorem 6.** *If  $\overline{\mathcal{R}_R} = E$ , then  $\mathcal{R}_{R'} = \mathcal{R}_{r(R')}$  if and only if  $\overline{\mathcal{R}_{R'}} = E'$ .*

*Proof.* Assume that  $\overline{\mathcal{R}_R} = E$ . We first prove the necessity part. Suppose that  $\mathcal{R}_{R'} = \mathcal{R}_{r(R')}$ . Fix  $\lambda \in \mathbb{R}_+^\bullet$  arbitrarily. If  $x' \in E'$ , then, in view of (13),  $R_\lambda(A')x'$  falls into  $\mathcal{R}_{R'}$ , and hence also into  $\mathcal{R}_{r(R')}$ . On account of (15) and (16),

$$(17) \quad \mathcal{R}_{r(R')} = \{R_\lambda(A')y' \mid y' \in \overline{\mathcal{R}_{R'}}\}.$$

Therefore there exists  $y' \in \overline{\mathcal{R}_{R'}}$  such that  $R_\lambda(A')x' = R_\lambda(A')y'$ . But  $R_\lambda(A')$  is one-to-one, so  $x' = y'$ , which proves that  $E' \subset \overline{\mathcal{R}_{R'}}$ . Since the reverse inclusion is obvious, we obtain  $\overline{\mathcal{R}_{R'}} = E'$ .

The proof of the sufficiency part is simple: If  $\overline{\mathcal{R}_{R'}} = E'$ , then the equality  $\mathcal{R}_{r(R')} = \mathcal{R}_{R'}$  is a consequence of (13) and (17).  $\square$

Bearing in mind (14), from Theorem 6 we immediately deduce the following result:

**Corollary 1.** *Let  $E$  be a Banach space, let  $S$  be a bounded strongly continuous one-parameter semigroup on  $E$  with the property that  $E^\circ \subsetneq E'$ , and let  $R$  be the resolvent of the generator of  $S$ , indexed by  $\mathbb{R}_+^\bullet$ . Then  $\mathcal{R}_{r(R')} \subsetneq \mathcal{R}_{R'}$ .*

We finally note that Banach spaces and semigroups satisfying the hypothesis of this corollary exist and are the subject of many expositions (see [3, p. 52–55], [8, p. 24–26], [12, p. 344–347]).

## 4. STRICT INCLUSIONS

In this section we present two examples showing that both inclusions in (5) may simultaneously be strict.

**Example 1.** Let  $C_b(\mathbb{R}_+)$  be the Banach space of all complex-valued bounded continuous functions on  $\mathbb{R}_+$ , and let  $C_{bu}(\mathbb{R}_+)$  be the Banach space of all complex-valued bounded uniformly continuous functions on  $\mathbb{R}_+$ . It is understood that both these spaces are endowed with the supremum norm. We regard  $C_{bu}(\mathbb{R}_+)$  as being isometrically embedded into  $C_b(\mathbb{R}_+)$ . Let  $L^\infty(\mathbb{R}_+)$  be the Banach space of (the equivalence classes of) Lebesgue measurable essentially bounded functions on  $\mathbb{R}_+$ , equipped with the norm  $\|x\| = \text{ess sup}_{t \in \mathbb{R}_+} |x(t)|$ .

For each  $t \in \mathbb{R}_+$ , let  $T_t^- : C_b(\mathbb{R}_+) \rightarrow C_b(\mathbb{R}_+)$  be the *backward shift* by  $t$  given by

$$(T_t^- f)(x) = f(x+t) \quad (f \in C_b(\mathbb{R}_+), x \in \mathbb{R}_+).$$

The family  $T^- = \{T_t^-\}_{t \in \mathbb{R}_+}$  is a semigroup of linear contractions on  $C_b(\mathbb{R}_+)$ . This semigroup is not strongly continuous. Nonetheless, it can be used to define a pseudo-resolvent in  $\mathcal{L}(C_b(\mathbb{R}_+))$ , tempered at infinity. Guided by (3), we let, for each  $\lambda \in \mathbb{R}_+^\bullet$ , each  $f \in C_b(\mathbb{R}_+)$ , and each  $x \in \mathbb{R}_+$ ,

$$(R_\lambda f)(x) = \int_0^\infty e^{-\lambda t} (T_t^- f)(x) dt = e^{\lambda x} \int_x^\infty e^{-\lambda t} f(t) dt.$$

Clearly, the function  $R_\lambda : x \mapsto (R_\lambda f)(x)$  is bounded and uniformly continuous; the mapping  $R_\lambda : f \mapsto R_\lambda f$  is a bounded linear operator on  $C_b(\mathbb{R}_+)$  of norm equal to  $\lambda^{-1}$ ; and the family  $R = \{R_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  is a pseudo-resolvent in  $\mathcal{L}(C_b(\mathbb{R}_+))$  with  $c_R = 1$ .

We claim that

$$(18) \quad \mathcal{R}_R = \{f \in C_{bu}(\mathbb{R}_+) \mid f' \in C_b(\mathbb{R}_+)\}.$$

Fix  $\lambda \in \mathbb{R}_+^\bullet$  arbitrarily and let  $f \in C_b(\mathbb{R}_+)$ . We have already noticed that  $R_\lambda f \in C_{bu}(\mathbb{R}_+)$ . It is also clear that  $R_\lambda f$  is differentiable, and that  $(R_\lambda f)' = \lambda R_\lambda f - f$ . Thus  $(R_\lambda f)'$  belongs to  $C_b(\mathbb{R}_+)$ , and so

$$\mathcal{R}_R \subset \{f \in C_{bu}(\mathbb{R}_+) \mid f' \in C_b(\mathbb{R}_+)\}.$$

To show the reverse inclusion, let  $f$  be a differentiable function in  $C_{bu}(\mathbb{R}_+)$  such that  $f' \in C_b(\mathbb{R}_+)$ . It is directly verified that  $R_\lambda(\lambda f - f') = f$ . Hence  $f \in \mathcal{R}_R$ , establishing the inclusion in question and thereby also proving (18).

It immediately follows from (18) that  $\overline{\mathcal{R}_R} = C_{bu}(\mathbb{R}_+)$ . Let  $S = \{S_t\}_{t \in \mathbb{R}_+}$  be the semigroup associated with  $R$ . It is clear that, for each  $t \in \mathbb{R}_+$ ,

$$S_t = T_t^- \upharpoonright C_{bu}(\mathbb{R}_+).$$

The generator of  $S$  is easily identified with the differentiation operator  $f \mapsto f'$  having  $\{f \in C_{bu}(\mathbb{R}_+) \mid f' \in C_b(\mathbb{R}_+)\}$  as a domain. Thus

$$\mathcal{R}_{r(R)} = \{f \in C_{bu}(\mathbb{R}_+) \mid f' \in C_b(\mathbb{R}_+)\}.$$

Finally, an application of Lebesgue's Differentiation Theorem reveals that

$$\text{Fav}(S) = \{f \in C_{bu}(\mathbb{R}_+) \mid f \text{ is differentiable } \lambda_{\mathbb{R}_+}\text{-a. e. and } f' \in L^\infty(\mathbb{R}_+)\}.$$

It is now clear that

$$\mathcal{R}_{r(R)} \subsetneq \mathcal{R}_R \subsetneq \text{Fav}(S).$$

**Example 2.** Let  $\mathfrak{B}(\mathbb{R}_+)$  be the  $\sigma$ -algebra of all Borel subsets of  $\mathbb{R}_+$ . Let  $M(\mathbb{R}_+)$  be the collection of all complex bounded regular Borel measures on  $\mathbb{R}_+$ . With addition and scalar multiplication defined pointwise on  $\mathfrak{B}(\mathbb{R}_+)$ , and with the norm  $\|\mu\| = |\mu|(\mathbb{R}_+)$ , where  $|\mu|$  denotes the total variation of the measure  $\mu \in M(\mathbb{R}_+)$ ,  $M(\mathbb{R}_+)$  is a Banach space. With the multiplication defined by the convolution of measures

$$(\mu * \nu)(A) = \int_{\mathbb{R}_+} \mu(A \ominus t) d\nu(t) \quad (A \in \mathfrak{B}(\mathbb{R}_+)),$$

where

$$A \ominus t = \{x \in \mathbb{R}_+ \mid x = y - t \text{ for some } y \in A\},$$

$M(\mathbb{R}_+)$  becomes a Banach algebra. The Dirac measure concentrated at 0,  $\delta_0$ , is the identity of this algebra.

For each  $f \in L^1(\mathbb{R}_+)$ , let  $\nu_f$  be a measure in  $M(\mathbb{R}_+)$  defined as

$$\nu_f(A) = \int_A f(t) dt \quad (A \in \mathfrak{B}(\mathbb{R}_+)).$$

The mapping  $f \mapsto \nu_f$  is a Banach algebra isomorphism of  $L^1(\mathbb{R}_+)$  onto the algebra of all measures in  $M(\mathbb{R}_+)$  that are absolutely continuous with respect to Lebesgue measure. We identify  $L^1(\mathbb{R}_+)$  with its image via  $f \mapsto \nu_f$ . Under this identification,  $L^1(\mathbb{R}_+)$  becomes an ideal of  $M(\mathbb{R}_+)$ . If  $f \in L^1(\mathbb{R}_+)$  and  $\mu \in M(\mathbb{R}_+)$ , then  $f * \mu$  is a member of  $L^1(\mathbb{R}_+)$  determined by  $\nu_f * \mu = \nu_{f*\mu}$  and given by

$$(19) \quad (f * \mu)(x) = \int_{[0,x]} f(x-t) d\mu(t) = \int_{\mathbb{R}_+} (\mathbb{T}_t^+ f)(x) d\mu(t) \quad (\lambda_{\mathbb{R}_+}\text{-a. e. } x \in \mathbb{R}_+).$$

Consider a homomorphism  $H: L^1(\mathbb{R}_+) \rightarrow \mathcal{L}(M(\mathbb{R}_+))$  defined by

$$H(f)\mu = f * \mu \quad (f \in L^1(\mathbb{R}_+), \mu \in M(\mathbb{R}_+)).$$

It is clear that  $\|H\| = 1$ . Since  $L^1(\mathbb{R}_+)$  is an ideal of  $M(\mathbb{R}_+)$ , we have  $\mathcal{R}_H \subset L^1(\mathbb{R}_+)$ . As  $H(f)\delta_0 = f$  for any  $f \in L^1(\mathbb{R}_+)$ , we see that

$$(20) \quad \mathcal{R}_H = L^1(\mathbb{R}_+).$$

For each  $\lambda \in \mathbb{R}_+^\bullet$ , set  $R_\lambda = H(\epsilon_{-\lambda})$ . Clearly,  $R = \{R_\lambda\}_{\lambda \in \mathbb{R}_+^\bullet}$  is a pseudo-resolvent in  $\mathcal{L}(M(\mathbb{R}_+))$  with  $c_R = 1$ . We shall show that

$$\mathcal{R}_{r(R)} \subsetneq \mathcal{R}_R \subsetneq \text{Fav}(S).$$

In view of (7) and (20), we have  $\overline{\mathcal{R}_R} = L^1(\mathbb{R}_+)$ . Using (19) (or alternatively utilising (8)), we verify at once that, for each  $\lambda \in \mathbb{R}_+^\bullet$  and each  $f \in L^1(\mathbb{R}_+)$ ,

$$(21) \quad r(R)_\lambda f = \int_{\mathbb{R}_+} e^{-\lambda t} \mathbb{T}_t^+ f dt.$$

Thus the semigroup associated with  $R$  coincides with the semigroup of forward shifts on  $L^1(\mathbb{R}_+)$ .

Given  $\mu \in M(\mathbb{R}_+)$  and  $s \in \mathbb{R}_+$ , denote by  $\mathcal{L}_\mu$  the Laplace transform of  $\mu$  defined as

$$\mathcal{L}_\mu(s) = \int_{\mathbb{R}_+} e^{-st} d\mu(t).$$

If  $f \in L^1(\mathbb{R}_+)$ , we write  $\mathcal{L}_f$  instead of  $\mathcal{L}_{\nu_f}$ . As is well known,  $\mathcal{L}_\mu$  uniquely determines  $\mu$ , and we also have  $\mathcal{L}_{\mu*\nu} = \mathcal{L}_\mu \mathcal{L}_\nu$  for any  $\mu, \nu \in M(\mathbb{R}_+)$ .

We proceed to show that the inclusion  $\mathcal{R}_{\mathcal{T}(R)} \subset \mathcal{R}_R$  is strict. We assert that, for each  $\lambda \in \mathbb{R}_+^\bullet$ ,

$$(22) \quad \epsilon_{-\lambda} \in \mathcal{R}_R \setminus \mathcal{R}_{\mathcal{T}(R)}.$$

Fix  $\lambda \in \mathbb{R}_+^\bullet$  arbitrarily. Since  $\epsilon_\lambda = R_\lambda(\delta_0)$ , we see that  $\epsilon_\lambda \in \mathcal{R}_R$ . Suppose, contrary to our assertion, that  $\epsilon_{-\lambda} \in \mathcal{R}_{\mathcal{T}(R)}$ . Then  $\epsilon_{-\lambda}$  can be represented as  $\epsilon_{-\lambda} = \epsilon_{-\lambda} * f$  for some  $f \in L^1(\mathbb{R}_+)$ . Accordingly,  $\mathcal{L}_{\epsilon_{-\lambda}} = \mathcal{L}_{\epsilon_{-\lambda}} \mathcal{L}_f$ . Since, for each  $s \in \mathbb{R}_+$ ,

$$\mathcal{L}_{\epsilon_{-\lambda}}(s) = \frac{1}{s + \lambda} \neq 0,$$

it follows that  $\mathcal{L}_f = 1 = \mathcal{L}_{\delta_0}$  and hence  $\nu_f = \delta_0$ . This contradiction establishes (22).

Let  $C(\mathbb{R}_+)$  be the space of all complex-valued (not necessarily bounded) continuous functions on  $\mathbb{R}_+$ . We claim that

$$(23) \quad \mathcal{R}_R \subset C(\mathbb{R}_+).$$

In fact, if we fix  $\lambda \in \mathbb{R}_+$  arbitrarily, then every element of  $\mathcal{R}_R$  can be represented as  $\epsilon_{-\lambda} * \mu$  for an appropriate  $\mu \in M(\mathbb{R}_+)$ . Clearly,

$$(\epsilon_{-\lambda} * \mu)(x) = e^{-\lambda x} \int_{[0,x)} e^{\lambda t} d\mu(t) \quad (\lambda_{\mathbb{R}_+}\text{-a. e. } x \in \mathbb{R}_+).$$

By Lebesgue's Dominated Convergence Theorem, the function

$$x \mapsto \int_{[0,x)} e^{\lambda t} d\mu(t)$$

is continuous. We thus see that  $\epsilon_{-\lambda} * \mu$  can be identified with a continuous function on  $\mathbb{R}_+$ . The claim is established.

We now show that the inclusion  $\mathcal{R}_R \subset \text{Fav}(\mathbb{T}^+)$  is strict. Choose arbitrarily  $a, b \in \mathbb{R}_+$  so that  $a < b$ . Set  $f = 1_{(a,b)}$ . Clearly,  $f$  belongs to  $L^1(\mathbb{R}_+)$ . We contend that  $f \in \text{Fav}(\mathbb{T}^+) \setminus \mathcal{R}_R$ . As usual, let  $A \triangle B$  denote the symmetric difference of the sets  $A$  and  $B$ ; that is

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Since, for any  $t \in \mathbb{R}_+$ ,

$$\|\mathbb{T}_t^+ f - f\| = \lambda_{\mathbb{R}_+}((a+t, b+t) \triangle (a, b)) \leq 2 \min(t, b-a),$$

we see that  $f \in \text{Fav}(\mathbb{T}^+)$ . Now note that the equivalence class of  $f$  in  $L^1(\mathbb{R}_+)$  has no representative that is continuous everywhere on  $\mathbb{R}_+$ . Indeed, if  $g$  were the (unique) continuous modification of  $f$ , then  $g$  would equal 1 on  $(a, b)$  and 0 off  $(a, b)$ , which is incompatible with the continuity of  $g$  on all of  $\mathbb{R}_+$ . Invoking (23), we see that  $f \notin \mathcal{R}_R$ . The contention is established.

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