FNS and HEIV: Relating Two Vision Parameter Estimation Frameworks

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Overview

- parameter estimation is a key problem in computer vision
- two recent frameworks for parameter estimation:
  - FNS (the authors)
  - HEIV (Meer's group)
- FNS and a core version of HEIV are essentially equivalent

Relevant Work

- Leedan and Meer (IJCV, 2000)
- Matei and Meer (CVPR, 2000)
- Matei (PhD, 2001)
- Chojnacki, Brooks, van den Hengel and Gawley
  - PAMI, 2000
  - PAMI, 2003
  - IVC, 2003

Set-up

- $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_l]^T$ — **parameter vector**
- $\boldsymbol{x} = [x_1, \ldots, x_k]^T$ — **data point**
- $\boldsymbol{u}(\boldsymbol{x}) = [u_1(\boldsymbol{x}), \ldots, u_l(\boldsymbol{x})]^T$ — **transformed data point**
  - **each component** $u_i(\boldsymbol{x})$ **is a quadratic form in** $[\boldsymbol{x}^T, 1]^T$
  - **one component is equal to** 1**.**
- ***principal*** **equation**

$$\boldsymbol{\theta}^T \boldsymbol{u}(\boldsymbol{x}) = 0$$

- ***ancillary*** **constraint**

$$\phi(\boldsymbol{\theta}) = 0 \qquad \phi(t\boldsymbol{\theta}) = t^\kappa \phi(\boldsymbol{\theta})$$

Algebraic Least Squares

- $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ **— data points**
- **the *algebraic least squares* cost function**

$$\boldsymbol{A} = \sum_{i=1}^{n} \boldsymbol{u}(\boldsymbol{x}_i)\boldsymbol{u}(\boldsymbol{x}_i)^T \qquad \|\boldsymbol{\theta}\| = (\theta_1^2 + \cdots + \theta_l^2)^{1/2}$$

$$J_{\mathrm{ALS}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) = \frac{\boldsymbol{\theta}^T \boldsymbol{A} \boldsymbol{\theta}}{\|\boldsymbol{\theta}\|^2}$$

- $\widehat{\boldsymbol{\theta}}_{\mathrm{ALS}} = \arg\min_{\boldsymbol{\theta} \neq \mathbf{0}} J_{\mathrm{ALS}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$
- $\widehat{\boldsymbol{\theta}}_{\mathrm{ALS}}$ **is an eigenvector of A associated with the smallest eigenvalue**

Approximated Maximum Likelihood Estimation

- ***approximated maximum likelihood*** **cost function**

$$J_{\mathrm{AML}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \dots, \boldsymbol{x}_n) = \sum_{i=1}^{n} \frac{\boldsymbol{\theta}^T \boldsymbol{u}(\boldsymbol{x}_i) \boldsymbol{u}(\boldsymbol{x}_i)^T \boldsymbol{\theta}}{\boldsymbol{\theta}^T \partial_{\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}_i) \boldsymbol{\Lambda}_{\boldsymbol{x}_i} \partial_{\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}_i)^T \boldsymbol{\theta}},$$

- $\boldsymbol{\Lambda}_{\boldsymbol{x}_i}$ **—** ***covariance matrix*****; describes the uncertainty of** $\boldsymbol{x}_i$
- **AML estimates**

$$\widehat{\boldsymbol{\theta}}^u_{\mathrm{AML}} = \arg\min_{\boldsymbol{\theta} \neq \mathbf{0}} J_{\mathrm{AML}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \qquad \textbf{unconstrained}$$

$$\widehat{\boldsymbol{\theta}}_{\mathrm{AML}} = \arg\min_{\substack{\phi(\boldsymbol{\theta})=0 \\ \boldsymbol{\theta} \neq \mathbf{0}}} J_{\mathrm{AML}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \qquad \textbf{constrained}$$

Algorithms for minimisation

- **unconstrained minimisation: FNS and HEIV**
- **constrained minimisation: CFNS**

Unconstrained minimisation

- **algebra**

$$[\partial_{\boldsymbol{\theta}} J_{\mathrm{AML}}(\boldsymbol{\theta}; \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)]^T = 2\boldsymbol{X}_{\boldsymbol{\theta}}\boldsymbol{\theta}$$

$$\boldsymbol{X}_{\boldsymbol{\theta}} = \sum_{i=1}^{n} \frac{\boldsymbol{A}_i}{\boldsymbol{\theta}^T \boldsymbol{B}_i \boldsymbol{\theta}} - \sum_{i=1}^{n} \frac{\boldsymbol{\theta}^T \boldsymbol{A}_i \boldsymbol{\theta}}{(\boldsymbol{\theta}^T \boldsymbol{B}_i \boldsymbol{\theta})^2} \boldsymbol{B}_i$$

$$\boldsymbol{A}_i = \boldsymbol{u}(\boldsymbol{x}_i)\boldsymbol{u}(\boldsymbol{x}_i)^T \qquad \boldsymbol{B}_i = \partial_{\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}_i) \boldsymbol{\Lambda}_{\boldsymbol{x}_i} \partial_{\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}_i)^T$$

- **variational equation**

$$\boldsymbol{X}_{\boldsymbol{\theta}}\boldsymbol{\theta} = \mathbf{0}$$

- **employ the *ordinary* eigenvalue problem**

$$\boldsymbol{X}_{\boldsymbol{\theta}}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$$

Fundamental numerical scheme

- **set θ to $\widehat{\theta}_{\mathrm{ALS}}$**
- **repeat:**
  - **compute the matrix $\boldsymbol{X}_{\boldsymbol{\theta}}$;**
  - **compute a normalised eigenvector of the eigenvalue problem**

$$\boldsymbol{X}_{\boldsymbol{\theta}}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$$

    corresponding to the eigenvalue closest to zero (in absolute value)
  - **take the computed eigenvector for an update of θ**

until convergence

Alternative form of the variational equation

- **split**

$$\boldsymbol{X}_{\boldsymbol{\theta}} = \boldsymbol{M}_{\boldsymbol{\theta}} - \boldsymbol{N}_{\boldsymbol{\theta}}$$

$$\boldsymbol{M}_{\boldsymbol{\theta}} = \sum_{i=1}^{n} \frac{\boldsymbol{A}_i}{\boldsymbol{\theta}^T \boldsymbol{B}_i \boldsymbol{\theta}} \qquad \boldsymbol{N}_{\boldsymbol{\theta}} = \sum_{i=1}^{n} \frac{\boldsymbol{\theta}^T \boldsymbol{A}_i \boldsymbol{\theta}}{(\boldsymbol{\theta}^T \boldsymbol{B}_i \boldsymbol{\theta})^2} \boldsymbol{B}_i$$

- **reformulate**

$$\boldsymbol{M}_{\boldsymbol{\theta}} \boldsymbol{\theta} = \boldsymbol{N}_{\boldsymbol{\theta}} \boldsymbol{\theta}$$

- **employ the *generalised* eigenvalue problem**

$$\boldsymbol{M}_{\boldsymbol{\theta}} \boldsymbol{\xi} = \lambda \boldsymbol{N}_{\boldsymbol{\theta}} \boldsymbol{\xi}$$

Basic HEIV Scheme

- **set θ to $\widehat{\theta}_{\mathrm{ALS}}$**
- **repeat:**
  - **compute the matrices M_{θ} and N_{θ}**
  - **compute a normalised eigenvector of the eigenvalue problem**

$$M_{\theta}\xi = \lambda N_{\theta}\xi$$

    corresponding to the eigenvalue closest to 1
  - **take the computed eigenvector for an update of θ**

until convergence

Reduced Variational Equation

- **problem: $N_{\boldsymbol{\theta}}$ is degenerate**
- **solution: reduce the variational equation**

$$\boldsymbol{u}(\boldsymbol{x}) = [\boldsymbol{z}(\boldsymbol{x})^T, 1]^T \qquad \boldsymbol{\theta} = [\boldsymbol{\eta}^T, \alpha]^T$$

$$\boldsymbol{B}_i = \begin{bmatrix} \boldsymbol{B}_i^0 & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix} \qquad \boldsymbol{B}_i^0 = \partial_{\boldsymbol{x}} \boldsymbol{z}(\boldsymbol{x}_i) \boldsymbol{\Lambda}_{\boldsymbol{x}_i} \partial_{\boldsymbol{x}} \boldsymbol{z}(\boldsymbol{x}_i)^T$$

$$\beta_i = \frac{1}{\boldsymbol{\eta}^T \boldsymbol{B}_i^0 \boldsymbol{\eta}} \qquad \tilde{\boldsymbol{z}} = \frac{\sum_{i=1}^n \beta_i \boldsymbol{z}_i}{\sum_{i=1}^n \beta_i} \qquad \boldsymbol{z}'_i = \boldsymbol{z}_i - \tilde{\boldsymbol{z}}$$

$$\boldsymbol{M}'_{\boldsymbol{\eta}} = \sum_{i=1}^n \beta_i \boldsymbol{z}'_i \boldsymbol{z}'^T_i \qquad \boldsymbol{N}'_{\boldsymbol{\eta}} = \sum_{i=1}^n \left(\beta_i \boldsymbol{z}'^T_i \boldsymbol{\eta}\right)^2 \boldsymbol{B}_i^0$$

Reduced Variational Equation (Cnt'd)

- **variational system**

$$\boldsymbol{M}'_{\boldsymbol{\eta}}\boldsymbol{\eta} = \boldsymbol{N}'_{\boldsymbol{\eta}}\boldsymbol{\eta}$$
$$\alpha = -\tilde{\boldsymbol{z}}^T\boldsymbol{\eta}$$

- **the system decouples**
- $\boldsymbol{N}'_{\boldsymbol{\eta}}$ **is generically *positive definite* if** $n \geq l$
- **two splits**

$$\widehat{\boldsymbol{\theta}}_{\mathrm{ALS}} = [(\widehat{\boldsymbol{\eta}}_{\mathrm{ALS}})^T, \widehat{\alpha}_{\mathrm{ALS}}]^T \qquad \widehat{\boldsymbol{\theta}}^u_{\mathrm{AML}} = [(\widehat{\boldsymbol{\eta}}^u_{\mathrm{AML}})^T, \widehat{\alpha}^u_{\mathrm{AML}}]^T$$

Reduced HEIV Scheme

- **set η to $\widehat{\boldsymbol{\eta}}_{\mathrm{ALS}}$**
- **repeat:**
  - **compute the matrices $M'_{\boldsymbol{\eta}}$ and $N'_{\boldsymbol{\eta}}$**
  - **compute a normalised eigenvector of the eigenvalue problem**

$$\boldsymbol{M}'_{\boldsymbol{\eta}}\boldsymbol{\zeta} = \lambda \boldsymbol{N}'_{\boldsymbol{\eta}}\boldsymbol{\zeta}$$

    corresponding to the eigenvalue closest to 1 and take this eigenvector for η
  - **take the computed eigenvector for an update of η**

until convergence

Conclusion

- **FNS and HEIV in basic and reduced forms are essentially equivalent**
- **FNS admits a constrained extension (CFNS)**
- **the above equivalence opens the way to extending the HEIV approach**