ON SOME HOLOMORPHIC DYNAMICAL SYSTEMS

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1. Introduction

LET D be a domain in $\mathbb{C}^N$ and φ a holomorphic automorphism of D. Let $\mathscr{C}$ be the measure class of the Lebesgue measure in D, i.e., the set of all positive regular Borel measures on D whose null sets coincide with the Lebesgue null sets. Let φ_* be the automorphism of $\mathscr{C}$ given by

$$(\varphi_*\mu)(B) = \mu(\varphi^{-1}(B)) \quad (\mu \in \mathscr{C},\ B \in \mathscr{B}(D)),$$

where $\mathscr{B}(D)$ denotes the Borel σ-algebra of D. Adopting the terminology introduced in [2], we will say that φ_* is finite if it has a fixed point among probability measures.

Let $L^2H(D)$ be the Hilbert space of all square Lebesgue integrable holomorphic functions on D. Suppose that $L^2H(D) \neq \{0\}$. Let U_φ be the unitary operator in $L^2H(D)$ defined by

$$U_\varphi f = (f \circ \varphi)J_\varphi \quad (f \in L^2H(D)),$$

where

$$J_\varphi = \det\left(\frac{\partial \varphi_i}{\partial z_j}\right)_{1 \le i,\, j \le N} \quad (\varphi = (\varphi_1, \ldots, \varphi_N)).$$

The purpose of this paper is to exhibit various relations between φ, φ_*, and U_φ. A fundamental result is that U_φ has either pure point spectrum or purely continuous spectrum, the first case occurring exactly when φ_* is finite. We prove that any of the following two conditions ensures the finiteness of φ_*: 1° the existence of a φ-invariant probability measure absolutely continuous with respect to Lebesgue measure; 2° the existence of a relatively compact orbit of φ. Of course, the first condition is also necessary. We show the necessity of a stronger version of the second condition (embeddability of φ in a compact transformation group) provided some mild restrictions on D are imposed. Assuming some hypotheses on D, we prove also that if a point in D is wandering, then U_φ has purely absolutely continuous spectrum, and, conversely, if U_φ has a non-zero absolutely continuous component in the spectrum, then all points in D are wandering. In particular, the spectrum of U_φ is either pure point, or purely absolutely continuous, or purely singular continuous. We show that if D is in a class of domains containing among

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others all bounded analytic polyhedra, then the spectrum of U_φ cannot be purely singular continuous.

2. A purity theorem

The starting point of our discussion is the following.

THEOREM 1.1. *Suppose there exists a φ-invariant probability measure $\mathbb{P}$ that is absolutely continuous with respect to Lebesgue measure. Then U_φ has pure point spectrum.*

Proof. Given $k \in \mathbb{Z}$ and a bounded Borel measure μ on the unit circle $\mathbb{T}$, let $\hat{\mu}(k)$ be the kth Fourier coefficient of μ, i.e.,

$$\hat{\mu}(k) = \int_{\mathbb{T}} t^{-k} \, d\mu(t).$$

By the spectral theorem for unitary operators, there is a unique projection-valued measure P on $\mathbb{T}$, taking values in a Boolean algebra of projections in $L^2H(D)$, such that for each $k \in \mathbb{Z}$,

$$U_\varphi^k = \int_{\mathbb{T}} t^k \, dP(t), \tag{1.1}$$

where the integral is to be interpreted in the sense of strong convergence. For any $f, g \in L^2H(D)$, let $\pi_{f,g}$ be the complex measure on $\mathbb{T}$ such that

$$\pi_{f,g}(B) = (P(B)f, g) \quad (B \in \mathscr{B}(\mathbb{T})),$$

where $(\cdot, \cdot)$ stands for the scalar product in $L^2H(D)$. Given $k \in \mathbb{N}$, let

$$\varphi^k = \underbrace{\varphi \circ \cdots \circ \varphi}_{k \text{ times}}, \qquad \varphi^{-k} = \underbrace{\varphi^{-1} \circ \cdots \circ \varphi^{-1}}_{k \text{ times}},$$

and let φ^0 stand for the identity map of D. Since $U_\varphi^k = U_{\varphi^k}$ for each $k \in \mathbb{Z}$, it follows from (1.1) that

$$(U_{\varphi^k} f, g) = \hat{\pi}_{f,g}(-k). \tag{1.2}$$

Let

$$L^2H(D) = H_{pp} \oplus H_c$$

be the orthogonal decomposition of $L^2H(D)$ in which H_{pp} is the closure of the linear span of the eigenvectors of U_φ, and H_c consists of those $f \in L^2H(D)$ for which $\pi_{f,f}$ is a continuous measure. As is known, the decomposition reduces P.

Suppose that the spectrum of U_φ is not pure point, that is, there exists a non-zero f in H_c. Let g be any element of $L^2H(D)$ with the decomposition $g = g' + g''$ with respect to H_{pp} and H_c. Clearly, $\pi_{f,g} = \pi_{f,g''}$. Since

$$\pi_{f,g''} = \tfrac{1}{4} \sum_{k=0}^{3} i^k \pi_{f+i^k g'', f+i^k g''}, \tag{1.3}$$

the measure $\pi_{f,g}$ is continuous. By a theorem of Wiener (cf. [**14**], p. 108),

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} |\hat{\pi}_{f,g}(k)|^2 = \sum_{t\in\mathbb{T}} |\pi_{f,g}(\{t\})|^2 = 0.$$

Thus in view of (1.2)

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} |(U_{\varphi^k} f, g)|^2 = 0. \tag{1.4}$$

Let $\{\varphi_n\colon n \in \mathbb{N}\}$ be any complete orthonormal set in $L^2H(D)$. The function

$$K_D(z, w) = \sum_{n=1}^{\infty} \varphi_n(z)\overline{\varphi_n(w)} \quad (z, w \in D), \tag{1.5}$$

called the Bergman function of D, does not depend on the particular choice of $\{\varphi_n\colon n \in \mathbb{N}\}$ (cf. [**1**], p. 21). For each $z \in D$, let χ_z be the element of $L^2H(D)$ such that

$$\chi_z(w) = K_D(w, z) \quad (w \in D). \tag{1.6}$$

A fundamental property of K_D is that, given $f \in L^2H(D)$ and $z \in D$,

$$f(z) = (f, \chi_z). \tag{1.7}$$

Let K be a compact subset of D such that $\mathbb{P}(K) > 0$. It is well known that

$$\sup\{\|\chi_z\|\colon z \in K\} \leq \pi^{-N/2} \operatorname{dist}(K, \partial D)^{-1}, \tag{1.8}$$

where $\|\cdot\|$ stands for the norm in $L^2H(D)$ and the distance $\operatorname{dist}(K, \partial D)$ between K and the boundary ∂D of D refers to the supremum norm in $\mathbb{C}^N$. From this estimate it follows that the restrictions to K of the functions

$$z \to \frac{1}{2n+1} \sum_{k=-n}^{n} |(U_{\varphi^k} f, \chi_z)|^2 \quad (n \in \mathbb{N})$$

are uniformly bounded. On account of (1.4), we find that

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} \int_K |(U_{\varphi^k} f, \chi_z)|^2 \, d\lambda(z) = 0, \tag{1.9}$$

where λ stands for the Lebesgue measure in D. By (1.7) for each $k \in \boldsymbol{Z}$,

$$\int_K |(U_{\varphi^k} f, \chi_z)|^2 \, d\lambda(z) = \int_K |(f \circ \varphi^k) J_{\varphi^k}|^2 \, d\lambda = \int_{\varphi^k(K)} |f|^2 \, d\lambda, \quad (1.10)$$

so if we denote by 1_B the characteristic function of a subset B of D, then (1.9) can be rewritten in the form

$$\lim_{n\to\infty} \int_D \left(\frac{1}{2n+1} \sum_{k=-n}^{n} 1_{\varphi^k(K)} \right) |f|^2 \, d\lambda = 0.$$

As $f \neq 0$ λ-a.e., the last identity and Fatou's lemma imply

$$\liminf_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} 1_{\varphi^k(K)} = 0 \quad \lambda\text{-a.e.} \quad (1.11)$$

Let $\mathfrak{M}$ be the σ-algebra $\{B \in \mathcal{B}(D)\colon\ 1_B = 1_{\varphi(B)}\ \mathbb{P}\text{-a.e.}\}$ and $\mathbb{E}^{\mathfrak{M}}$ be the corresponding conditional expectation operator. By Birkhoff's ergodic theorem (cf. [**11**], p. 25),

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} 1_{\varphi^k(K)} = \mathbb{E}^{\mathfrak{M}}(1_K) \quad \mathbb{P}\text{-a.e.}$$

Comparing this equality with (1.11) and taking into account that $\mathbb{P}$ is absolutely continuous with respect to λ, we find that

$$\mathbb{E}^{\mathfrak{M}}(1_K) = 0 \quad \mathbb{P}\text{-a.e.}$$

Hence

$$\mathbb{P}(K) = \mathbb{E}[\mathbb{E}^{\mathfrak{M}}(1_K)] = 0,$$

a contradiction.

The proof is complete.

THEOREM 1.2. *If U_φ has a non-zero eigenvector, then φ_* is finite.*

Proof. Let h be an eigenvector of U_φ of unit norm. Setting

$$\mathbb{P}(B) = \int_B |h|^2 \, d\lambda \quad (B \in \mathcal{B}(D))$$

defines a φ-invariant probability measure on D. Since $h \neq 0$ λ-a.e., it follows that $\mathbb{P} \in \mathscr{C}$.

The proof is complete.

As a corollary to Theorems 1.1 and 1.2, we obtain the following generalization of a result of [**8**].

THEOREM 1.3. *The spectrum of U_φ is either pure point or purely continuous according as φ_* is finite or not.*

We close this section with two simple examples.

EXAMPLE 1.4. Let D be an open ball in $\mathbb{C}^N$ centered at 0 and let φ be defined as

$$\varphi(z_1, \ldots, z_N) = (e^{i\theta_1}z_1, \ldots, e^{i\theta_N}z_N) \quad (z_1, \ldots, z_N \in \mathbb{C}),$$

where $\theta_1, \ldots, \theta_N \in \mathbb{R}$. Then the normalized Lebesgue measure in D is φ-invariant, and so, by Theorem 1.3, U_φ has pure point spectrum.

EXAMPLE 1.5. Let $D = \{(z_1, \ldots, z_N) \in \mathbb{C}^N\colon \operatorname{Im} z_1 > 0, \ldots, \operatorname{Im} z_N > 0\}$ and let φ be defined as

$$\varphi(z_1, \ldots, z_N) = (z_1 + t_1, \ldots, z_N + t_N) \quad (z_1, \ldots, z_N \in \mathbb{C}),$$

where $(t_1, \ldots, t_n) \in \mathbb{R}^N \setminus \{(0, \ldots, 0)\}$. Let U be an open ball in $\mathbb{C}^N$ with sufficiently small radius so that $\varphi^m(U) \cap \varphi^n(U) = \emptyset$ for any distinct integers m and n. Suppose that there exists a ϕ-invariant probability measure $\mathbb{P} \in \mathscr{C}$. If $\mathscr{N}$ is any finite subset of $\mathbb{Z}$ with cardinality n, then

$$n \cdot \mathbb{P}(U) = \sum_{m \in \mathscr{N}} \mathbb{P}(\varphi^m(U)) = \mathbb{P}\Big(\bigcup_{m \in \mathscr{N}} \varphi^m(U)\Big) \leq 1.$$

Hence $\mathbb{P}(U) = 0$, which is incompatible with $\mathbb{P} \in \mathscr{C}$. Therefore φ_* is not finite, and consequently, by Theorem 1.3, U_φ has purely continuous spectrum.

2. Pure point spectrum

For each $w \in D$, let $\mathcal{O}(w)$ denote the orbit $\{\varphi^k(w)\colon k \in \mathbb{Z}\}$ of w.

The following theorem generalizes some results of **[8, 9, 10]**.

THEOREM 2.1. *Suppose that there exists $w \in D$ such that the closure of $\mathcal{O}(w)$ in D is compact. Then U_φ has pure point spectrum.*

Proof. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$, let $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and, given $t \in D$, let $\bar\partial^\alpha \chi_t$ or $\partial_{\bar z_1^{\alpha_1} \ldots \bar z_N^{\alpha_N}} \chi_t$ denote the function

$$w \to \frac{\partial^{|\alpha|} K_D}{\partial \bar z_1^{\alpha_1} \cdots \partial \bar z_N^{\alpha_N}}(w, z)\bigg|_{z=t}.$$

It is easily seen that for each $\alpha \in (\mathbb{N} \cup \{0\})^N$ and each $z \in D$, $\bar\partial^\alpha \chi_z$ is in $L^2H(D)$ and

$$\frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_N^{\alpha_N}}(z) = (f, \bar\partial^\alpha \chi_z) \quad (f \in L^2H(D)).$$

Since $L^2H(D) \neq \{0\}$, it follows immediately from the last equality that for each $z \in D$ there is a multiindex α such that $\bar\partial^\alpha \chi_z \neq 0$.

Let α be a multiindex such that $\bar{\partial}^\alpha \chi_w \neq 0$ and $\bar{\partial}^\beta \chi_w = 0$ for every multiindex β with $|\beta| < |\alpha|$. Then

$$U_{\varphi^{-1}}(\bar{\partial}^\alpha \chi_w)$$
$$= \sum \partial_{\bar{z}_{i_1} \cdots \bar{z}_{i_{\alpha_1}} \bar{z}_{i_{\alpha_1+1}} \cdots \bar{z}_{i_{\alpha_1+\alpha_2}} \cdots} \chi_{\varphi(w)}$$
$$\times \overline{\frac{\partial \varphi_{i_1}}{\partial z_1}}(w) \cdot \cdots \cdot \overline{\frac{\partial \varphi_{i_{\alpha_1}}}{\partial z_1}}(w) \cdot \overline{\frac{\partial \varphi_{i_{\alpha_1+1}}}{\partial z_2}}(w) \cdot \cdots \cdot \overline{\frac{\partial \varphi_{i_{\alpha_1+\alpha_2}}}{\partial z_2}}(w) \cdot \cdots$$
$$\cdot \overline{J_\varphi(w)},$$

where the sum extends over all multiindices $(i_1, \ldots, i_{|\alpha|})$ with $1 \leqslant i_1 \leqslant N, \ldots, 1 \leqslant i_{|\alpha|} \leqslant N$. A similiar formula is valid with φ replaced by φ^k $(k \in \mathbb{Z} \setminus \{0\})$, α replaced by γ such that $|\gamma| = |\alpha|$, and w replaced by z in $\mathcal{O}(w)$. In particular, it follows from that formula that the dimension $d(z)$ of the linear span H_z of $\{\bar{\partial}^\gamma \chi_z \colon |\gamma| = |\alpha|\}$ takes on a constant value d for $z \in \mathcal{O}(w)$. Obviously, $d(z) \leqslant d$ for all z in the closure $\overline{\mathcal{O}(w)}$ of $\mathcal{O}(w)$ in D, and, since $\overline{\mathcal{O}(z)} = \overline{\mathcal{O}(w)}$ for each $z \in \mathcal{O}(w)$, we actually have $d(z) = d$ for all $z \in \overline{\mathcal{O}(w)}$. Now it is clear that $z \to H_z$ is a continuous mapping from $\overline{\mathcal{O}(w)}$ into the Grassmannian of d-dimensional linear subspaces of $L^2H(D)$, and that the image of $\overline{\mathcal{O}(w)}$ by that mapping is compact. Since, for each $k \in \mathbb{Z}$, $U_{\varphi^k}(\bar{\partial}^\alpha \chi_w)$ is an element of $H_{\varphi^{-k}(w)}$ of norm $\|\bar{\partial}^\alpha \chi_w\|$, it follows that the closure of $\{U_{\varphi^k}(\bar{\partial}^\alpha \chi_w) \colon k \in \mathbb{Z}\}$ is compact. Let H be the closure of the linear span of $\{U_{\varphi^k}(\bar{\partial}^\alpha \chi_w) \colon k \in \mathbb{Z}\}$. H is a reducing subspace for U_φ and, by Weyl's theorem (cf. [7], p. 456), the unitary representation $k \to U_{\varphi^k} \mid H$ of $\mathbb{Z}$ in H is a direct sum of finite dimensional unitary representations (here $U_{\varphi^k} \mid H$ stands for the restriction of U_{φ^k} to H). Since a unitary operator in a finite dimensional complex Hilbert space has pure point spectrum, it follows that U_φ has an eigenvector. By Theorem 1.3, U_φ actually has pure point spectrum.

The proof is complete.

3. Property (A)

We shall say that (D, φ) has property (A) if for each $w \in D$, there exist relatively compact open neighbourhoods U and V of w in D such that if $U \cap \varphi^n(U) \neq \emptyset$ for some $n \in \mathbb{Z}$, then $\varphi^n(U) \subset V$.

THEOREM 3.1. *Suppose that D is a domain in $\mathbb{C}$ and that U_φ has pure point spectrum. Then (D, φ) has property* (A).

Proof. Since $L^2H(D) \neq \{0\}$, it follows from a result of Wiegerinck [13] that $L^2H(D)$ is in fact infinite dimensional. In particular, there exist two linearly independent eigenvectors h_1, h_2 of U_φ. Let w be a point in D.

Clearly, at least one of the meromorphic functions h_1/h_2 and h_2/h_1, say h_1/h_2, is holomorphic at w. Let D' denote D with the poles of h_1/h_2 deleted. For any $\delta > 0$, let

$$W_\delta = \{z \in D' \colon |h_1(z)/h_2(z) - h_1(w)/h_2(w)| < \delta\},$$

and let V_δ be the (open) component of W_δ containing w. As is well known (cf. [4] p. 10), there exist $k \in \mathbb{N}$, an open neighbourhood $\Omega' \subset D'$ of w, an open neighbourhood Δ of 0, and one-to-one holomorphic functions $g_1\colon \Omega' \to \Delta$ and $g_2\colon \Delta \to \Delta$ such that

$$h_1(z)/h_2(z) - h_1(w)/h_2(w) = g_2([g_1(z)]^k)$$

for each $z \in \Omega'$. It follows from this representation that there exists $\delta > 0$ such that the closure of $V_{3\delta}$ in D is compact. The proof will be complete once we show that if $V_\delta \cap \varphi^n(V_\delta) \neq \emptyset$ for some $n \in \mathbb{Z}$, then $\varphi^n(V_\delta) \subset V_{3\delta}$.

Suppose that $\varphi^n(z_1) = z_2$ for $z_1, z_2 \in V_\delta$ and $n \in \mathbb{Z}$. Then, for each $z \in V_\delta$,

$$\begin{aligned}
&|h_1(\varphi^n(z))/h_2(\varphi^n(z)) - h_1(w)/h_2(w)| \\
&\quad \leq |h_1(\varphi^n(z))/h_2(\varphi^n(z)) - h_1(\varphi^n(z_1))/h_2(\varphi^n(z_1))| \\
&\qquad + |h_1(z_2)/h_2(z_2) - h_1(w)/h_2(w)| \\
&\quad = |h_1(z)/h_2(z) - h_1(z_1)/h_2(z_1)| + |h_1(z_2)/h_2(z_2) - h_1(w)/h_2(w)| < 3\delta.
\end{aligned}$$

Thus $\varphi^n(V_\delta) \subset W_{3\delta}$. Since $\varphi^n(V_\delta)$ is connected and intersects $V_\delta \subset V_{3\delta}$, it follows that $\varphi^n(V_\delta) \subset V_{3\delta}$.

The proof is complete.

We shall say that a domain Ω in $\mathbb{C}^N$ has property $\mathscr{A}$ if, for each $w \in \Omega$, $\chi_w \neq 0$ and there is $1 \leq i \leq N$ such that $\partial_{\bar{z}_i}\chi_w$ and χ_w are linearly independent.

THEOREM 3.2. *Suppose that D has property $\mathscr{A}$ and that U_φ has pure point spectrum. Then (D, φ) has property* (A).

Proof. Let w be a point in D. As $\chi_w \neq 0$, formulae (1.5) and (1.6) applied to a complete orthonormal set in $L^2H(D)$ consisting of eigenvectors of U_φ show that $h(w) \neq 0$ for some eigenvector h of U_φ. Let D' denote D with the zeros of h deleted. For any $\delta > 0$, let

$$W_\delta = \{z \in D' \colon \|(\overline{h(z)})^{-1}\chi_z - (\overline{h(w)})^{-1}\chi_w\| < \delta\},$$

and let V_δ be the component of W_δ containing w. Since for some $1 \leq i \leq N$, $\partial_{\bar{z}_i}\chi_w$ and χ_w are linearly independent, an open neighbourhood of w can be diffeomorphically embedded in $L^2H(D)$ by the map $z \to (\overline{h(z)})^{-1}\chi_z$. Accordingly, we can find $\delta > 0$ such that the closure of $V_{3\delta}$ in D is compact. The proof will be complete once we show that if $V_\delta \cap \varphi^n(V_\delta) \neq \emptyset$ for some $n \in \mathbb{Z}$, then $\varphi^n(V_\delta) \subset V_{3\delta}$.

Suppose that $\varphi^n(z_1) = z_2$ for $z_1, z_2 \in V_\delta$ and $n \in \mathbb{Z}$. Then, for each $z \in V_\delta$,

$$\|(\overline{h(\varphi^n(z))})^{-1}\chi_{\varphi^n(z)} - (\overline{h(w)})^{-1}\chi_w\| \leq \|U_{\varphi^n}[(\overline{h(z)})^{-1}\chi_z - (\overline{h(z_1)})^{-1}\chi_{z_1}]\| + \|(\overline{h(z_2)})^{-1}\chi_{z_2} - (\overline{h(w)})^{-1}\chi_w\| < 3\delta.$$

Thus $\varphi^n(V_\delta) \subset W_{3\delta}$. Since $\varphi^n(V_\delta)$ is connected and intersects $V_\delta \subset V_{3\delta}$, it follows that $\varphi^n(V_\delta) \subset V_{3\delta}$.

The proof is complete.

We shall say that a domain Ω in $\mathbb{C}^N$ has property $\mathscr{B}$ if, for each $w \in \Omega$, $\chi_w \neq 0$ (or, equivalently, $K_\Omega(w, w) > 0$) and the Bergman metric tensor

$$g_\Omega(w) = \sum_{i,j=1}^{N} \frac{\partial^2}{\partial z_i \, \partial \bar{z}_j} \log K_\Omega(w, w) \, \mathrm{d}z_i \, \mathrm{d}\bar{z}_j$$

is positive-definite. As is known (cf. [5], p. 296), every domain in $\mathbb{C}^N$ which is biholomorphically equivalent to a bounded domain has property $\mathscr{B}$.

If a domain Ω in $\mathbb{C}^N$ has property $\mathscr{B}$, then it has also property $\mathscr{A}$. In fact, if $\partial_{\bar{z}_j}\chi_w = \alpha_j\chi_w$ for $w \in \Omega$ and $\alpha_j \in \mathbb{C}$ $(1 \leq j \leq N)$, then

$$\frac{\partial}{\partial \bar{z}_j} K_\Omega(w, w) = \alpha_j K_\Omega(w, w)$$

and, for $1 \leq i \leq N$,

$$\frac{\partial^2}{\partial z_i \, \partial \bar{z}_j} K_\Omega(w, w) = \alpha_j \frac{\partial}{\partial z_i} K_\Omega(w, w)$$

whence

$$\frac{\partial^2}{\partial z_i \, \partial \bar{z}_j} \log K_\Omega(w, w) = -K_\Omega^{-2}(w, w) \cdot \frac{\partial}{\partial z_i} K_\Omega(w, w) \cdot \frac{\partial}{\partial \bar{z}_j} K_\Omega(w, w) + K_\Omega^{-1}(w, w) \cdot \frac{\partial^2}{\partial z_i \, \partial \bar{z}_j} K_\Omega(w, w) = 0.$$

THEOREM 3.3. *Suppose that D has property $\mathscr{B}$. Then (D, φ) has property* (A).

Proof. Let d_D be the geodesic distance relative to g_D. Let w be a point in D. For each $\delta > 0$, let $B_D(w, \delta)$ denote the open ball relative to d_D centered at w with radius δ. It is an elementary result from differential geometry that there is $\delta > 0$ such that the closure of $B_D(w, 3\delta)$ in D is compact. To end the proof, it suffices to show that if

$$\varphi^n(B_D(w, \delta)) \cap B_D(w, \delta) \neq \varnothing$$

for some $n \in \mathbb{Z}$, then

$$\varphi^n(B_D(w, \delta)) \subset B_D(w, 3\delta).$$

Suppose that $\varphi^n(z_1) = z_2$ for $z_1, z_2 \in B_D(w, \delta)$ and $n \in \mathbb{Z}$. As is well known (cf. [5], p. 299), φ is an isometry with respect to d_D. Therefore, for each $z \in B_D(w, \delta)$,

$$\begin{aligned} d_D(\varphi^n(z), w) &\leq d_D(\varphi^n(z), \varphi^n(z_1)) + d_D(z_2, w) \\ &= d_D(z, z_1) + d_D(z_2, w) < 3\delta, \end{aligned}$$

which yields the desired conclusion.

4. Pure point spectrum (continued)

Let $\mathcal{H}(D)$ be the space of all holomorphic mappings of D into itself equipped with the compact open topology. With composition as semigroup operation, $\mathcal{H}(D)$ is a topological semigroup. The identity mapping is the identity of $\mathcal{H}(D)$.

THEOREM 4.1. *Suppose that (D, φ) has property* (A) *and that U_φ has pure point spectrum. Then the closure of $\{\varphi^k\colon k \in \mathbb{Z}\}$ in $\mathcal{H}(D)$ is a compact topological group.*

Proof. We first show that if the closure G of $\{\varphi^k\colon k \in \mathbb{Z}\}$ in $\mathcal{H}(D)$ is compact, then it is a topological group. It is clear that G is a topological semigroup. Let ψ be an element of G. Since the topology of $\mathcal{H}(D)$ is metrisable, there exists a sequence $(k_n)_{n \in \mathbb{N}}$ in $\mathbb{Z}$ such that $\psi = \lim_{n\to\infty} \varphi^{k_n}$. By the compactness of G, there is a subsequence $(k_{n_m})_{m \in \mathbb{N}}$ of $(k_n)_{n \in \mathbb{N}}$ such that the sequence $(\varphi^{-k_{n_m}})_{m \in \mathbb{N}}$ converges to an element of G. Of course, $\lim_{m\to\infty} \varphi^{-k_{n_m}} = \psi^{-1}$. Thus G is a group. Since G is a compact group which is a topological semigroup, it follows from a theorem of Ellis [3] that G is a topological group.

Suppose now that G is not compact. Then, by Montel's theorem, there exists $z \in D$ such that for each open neighbourhood Ω of z, the closure of $\bigcup_{k \in \mathbb{Z}} \varphi^k(\Omega)$ in D is not compact. Let U and V be relatively compact neighbourhoods of z in D such that if $U \cap \varphi^n(U) \neq \emptyset$ for some $n \in \mathbb{Z}$, then $\varphi^n(U) \subset V$. An easy exhaustion argument shows the existence of a maximal subset $\mathcal{M}$ of $\mathbb{Z}$ such that $\varphi^m(U) \cap \varphi^n(U) = \emptyset$ for any distinct m and n in $\mathcal{M}$. We claim that $\mathcal{M}$ is infinite.

In fact, given $k \in \mathbb{Z}$, choose $m \in \mathcal{M}$ so that $\varphi^m(U) \cap \varphi^k(U) \neq \emptyset$. Then $U \cap \phi^{k-m}(U) \neq \emptyset$ whence $\varphi^{k-m}(U) \subset V$ and further $\varphi^k(U) \subset \varphi^m(V)$. Consequently,

$$\bigcup_{k \in \mathbb{Z}} \varphi^k(U) \subset \bigcup_{m \in \mathcal{M}} \varphi^m(V)$$

and as the closure of $\bigcup_{k \in \mathbb{Z}} \varphi^k(U)$ in D is not compact, $\mathcal{M}$ must be infinite, as claimed.

Applying now an argument from Example 1.5, we see that φ_* is not finite. Hence, in view of Theorem 1.3, the spectrum of U_φ cannot be pure point. This contradiction completes the proof.

As a corollary to Theorems 3.1, 3.2, and 4.1, we obtain

THEOREM 4.2. *Let D be either a domain in $\mathbb{C}$ or a domain in $\mathbb{C}^N$ $(N \geq 2)$ having property $\mathcal{A}$. Assume that U_φ has pure point spectrum. Then the closure of $\{\varphi^k: k \in \mathbb{Z}\}$ in $\mathcal{H}(D)$ is a compact topological group.*

Theorem 4.2 generalizes some results of [**8**, **9**]. The proofs to the theorems of which Theorem 4.2 is a consequence rely on a modification of an argument due to T. Mazur [**9**]. Mazur's original argument used differential geometry and involved an assumption on D stronger than property $\mathcal{B}$, namely, property $\mathcal{C}$ which will be introduced below.

As a consequence of Theorems 1.3, 2.1, and 4.1, we obtain

THEOREM 4.3. *Suppose that (D, φ) has property* (A). *If the orbit of some point in D has compact closure in D, then the orbits of all points in D have compact closure in D.*

The above theorem fails if all the assumptions about D and φ are dropped. This is shown by the following

EXAMPLE 4.4. Let $D = \mathbb{C}^2$ and let φ be defined as

$$\varphi(z_1, z_2) = (z_1, z_1 + z_2) \quad (z_1, z_2 \in \mathbb{C}).$$

Then all elements of $\{0\} \times \mathbb{C}$ are fixed points for φ, and for each $w \in \mathbb{C} \setminus (\{0\} \times \mathbb{C})$, the closure of $\mathcal{O}(\omega)$ is not compact.

5. Absolutely continuous spectrum

We shall say that a point w in D is wandering if there exists an open neighbourhood U of w such that $U \cap \varphi^n(U) = \emptyset$ for each $n \in \mathbb{N}$. Of course, we may equally well assume in this definition that $U \cap \varphi^n(U) = \emptyset$ holds for all $n \in \mathbb{Z} \setminus \{0\}$.

Suppose that $w \in D$ is not wandering, and let U be an open neighbourhood of w. Then, $U \cap \varphi^n(U) \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. In fact, if there is $n_0 \in \mathbb{N}$ such that $U \cap \varphi^n(U) = \emptyset$ for $n \geq n_0$, then $w \neq \varphi^n(w)$ for $n \geq n_0$ and hence for all $n \in \mathbb{N}$. Now, if V is an open neighbourhood of w contained in U such that $V \cap \varphi^n(V) = \emptyset$ for $1 \leq n \leq n_0$, then, clearly, $V \cap \varphi^n(V) = \emptyset$ for all $n \in \mathbb{N}$, a contradiction.

Let

$$H_c = H_{ac} \oplus H_{sc}$$

be the orthogonal decomposition of H_c in which H_{ac} consists of those $f \in H_c$ for which the measure $\pi_{f,f}$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{T}$, and H_{sc} consists of those $f \in H_c$ for which the measure $\pi_{f,f}$ is singular with respect to the Lebesgue measure in $\mathbb{T}$. As is known, the decomposition reduces the projection-valued measure P.

THEOREM 5.1. *Suppose that a point w in D is wandering. Then $\chi_w \in H_{ac}$.*

Proof. Let U be an open neighbourhood of w such that $U \cap \varphi^n(U) = \emptyset$ for all $n \in \mathbb{Z}\setminus\{0\}$. Then, clearly, $\phi^m(U) \cap \varphi^n(U) = \emptyset$ for distinct m and n in $\mathbb{Z}$.

By (1.10) and Levi's monotone convergence theorem,

$$\int_U \sum_{k\in\mathbb{Z}} |(U_{\varphi^k}\chi_w, \chi_z)|^2 \, d\lambda(z) = \sum_{k\in\mathbb{Z}} \int_{\varphi^k(U)} |\chi_w(z)|^2 \, d\lambda(z) \leq \|\chi_w\|^2.$$

This jointly with (1.2) and Plancherel's theorem shows that for λ-almost all z in U, the measure π_{χ_w,χ_z} is absolutely continuous with respect to the Lebesgue measure in $\mathbb{T}$. Since the mapping $D \ni z \to \chi_z \in L^2H(D)$ is continuous, it follows that for all z in U, the measure π_{χ_w,χ_z} is absolutely continuous with respect to the Lebesgue measure in $\mathbb{T}$. In particular, this is the case of π_{χ_w,χ_w}.

The proof is complete.

THEOREM 5.2. *If there is a wandering point in D, then $H_{ac} \neq \{0\}$.*

Proof. The set W of wandering points in D is clearly open. If $W \neq \emptyset$, then, in view of (1.6), there exists $w \in W$ such that $\chi_w \neq 0$. Now the theorem follows upon applying Theorem 5.1.

THEOREM 5.3. *Suppose that (D, φ) has property* (A) *and $H_{ac} \neq \{0\}$. Then every point in D is wandering.*

Proof. Suppose that $f \in H_{ac}\setminus\{0\}$. Let w be a point in D. Let U and V be open neighbourhoods of w with compact closure in D such that if $U \cap \varphi^n(U) \neq \emptyset$ for some $n \in \mathbb{Z}$, then $\varphi^n(U) \subset V$. Given $z \in V$, let $\chi_z = g_z' + g_z''$ be the decomposition of χ_z with respect to $H_{pp} \oplus H_{sc}$ and H_{ac}. Clearly, $\pi_{f,\chi_z} = \pi_{f,g_z''}$. Hence, in view of (1.3), the measure π_{f,χ_z} is absolutely continuous with respect to the Lebesgue measure in $\mathbb{T}$. By (1.2) and the Riemann–Lebesgue lemma,

$$\lim_{k\to\pm\infty} (U_{\varphi^k}f, \chi_z) = 0. \tag{5.1}$$

In view of (1.8), the restrictions to V of the functions $z \to (U_{\varphi^k}f, \chi_z)$ $(k \in \mathbb{Z})$ are uniformly bounded. Applying Lebesgue's dominated convergence theorem and taking into account (1.10) and (5.1), we get

$$\lim_{k\to\pm\infty} \int_{\varphi^k(V)} |f|^2 \, d\lambda = \lim_{k\to\pm\infty} \int_V |(U_{\varphi^k} f, \chi_z)|^2 \, d\lambda = 0. \tag{5.2}$$

Suppose that w is not wandering. Then there exists a sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers such that $\lim_{n\to\infty} k_n = +\infty$ and $U \cap \varphi^{k_n}(U) \neq \emptyset$ for each $n \in \mathbb{N}$. It is clear that given $n \in \mathbb{N}$, we have $\varphi^{k_n}(U) \subset V$. Hence $U \subset \varphi^{-k_n}(V)$ and further

$$\int_U |f|^2 \, d\lambda \leq \int_{\varphi^{-k_n}(V)} |f|^2 \, d\lambda.$$

The last inequality together with (5.2) yields

$$\int_U |f|^2 \, d\lambda = 0.$$

Since $f \neq 0$ λ-a.e. and $\lambda(U) > 0$, we get a contradiction.
The proof is complete.

THEOREM 5.4. *Suppose that (D, φ) has property* (A) *and that U_φ has a non-zero absolutely continuous component in the spectrum. Then the spectrum of U_φ is purely absolutely continuous.*

Proof. Suppose that $H_{ac} \neq \{0\}$. Then, by Theorem 5.3, every point in D is wandering, and, by Theorem 5.1, $\chi_w \in H_{ac}$ for each $w \in D$. In view of (1.7), the Hahn–Banach theorem, and the Riesz theorem, the linear span of $\{\chi_w : w \in D\}$ is dense in $L^2H(D)$. Therefore H_{ac} coincides with $L^2H(D)$.
The proof is complete.

As a consequence of Theorems 5.2, 5.3, and 5.4, we obtain

THEOREM 5.5. *Suppose that (D, φ) has property* (A). *If some point in D is wandering, then all points in D are so.*

Notice that the above theorem fails if all the assumptions about D and φ are dropped. In fact, if D and φ are as in Example 4.4, then the set of wandering points in D coincides with $\mathbb{C}^2 \setminus (\{0\} \times \mathbb{C})$.

We shall say that a domain Ω has property $\mathscr{C}$ if it has property $\mathscr{B}$ and is complete with respect to the geodesic distance d_Ω. As is known (cf. [**6**]), every domain in $\mathbb{C}^N$ which is biholomorphically equivalent to a bounded analytic polyhedron has property $\mathscr{C}$.

THEOREM 5.6. *Suppose that D has property $\mathscr{C}$. Then the spectrum of U_φ is either pure point or purely absolutely continuous.*

Proof. Let $\mathfrak{H}(D)$ be the group of all holomorphic automorphisms of D, the group operation being composition of mappings. Since D has property $\mathscr{C}$ and in particular has property $\mathscr{B}$, it follows that under the compact open topology, $\mathfrak{H}(D)$ is a Lie group (cf. [5], p. 300). Let G be the closure of $\{\varphi^n\colon n \in \mathbb{Z}\}$ in $\mathfrak{H}(D)$. Clearly, G is a locally compact monothetic group. Therefore, either G is compact or $G = \{\varphi^n\colon n \in \mathbb{Z}\}$ and $\mathbb{Z} \ni n \to \varphi^n \in G$ is a topological isomorphism of $\mathbb{Z}$ and G (cf. [12], p. 39). In the first case, by virtue of Theorem 2.1, U_φ has pure point spectrum. In the other case, it turns out that U_φ has purely absolutely continuous spectrum.

Indeed, suppose that $\{\varphi^n\colon n \in \mathbb{Z}\}$ is topologically isomorphic to $\mathbb{Z}$. If the spectrum of U_φ is not purely absolutely continuous, then, by Theorems 5.2 and 5.4, no point in D is wandering. Fix $w \in D$. Since D is complete with respect to d_D, it follows from the Hopf–Rinov theorem (cf. [5], p. 56) that for each $\delta > 0$, the closure of $B_D(w, \delta)$ in D is compact. Choose any $\delta_0 > 0$. Since w is not wandering, there is an unbounded sequence $(k_n)_{n \in \mathbb{N}}$ of positive integers such that

$$B_D(w, \delta_0) \cap \varphi^{k_n}(B_D(w, \delta_0)) \neq \emptyset$$

for each $n \in \mathbb{N}$. Consequently, for each $\delta \geq \delta_0$ and each $n \in \mathbb{N}$,

$$B_D(w, \delta) \cap \varphi^{k_n}(B_D(w, \delta)) \neq \emptyset$$

whence, as in the proof of Theorem 3.3,

$$\varphi^{k_n}(B_D(w, \delta)) \subset B_D(w, 3\delta).$$

By Montel's theorem and the fact that each compact subset of D is contained in some $B_D(w, \delta)$ $(\delta \geq \delta_0)$, we conclude that the closure of $\{\phi^{k_n}\colon n \in \mathbb{N}\}$ in $\mathfrak{H}(D)$ is compact. Since $\{\phi^n\colon n \in \mathbb{Z}\}$ is topologically isomorphic to $\mathbb{Z}$, this cannot be the case unless $(k_n)_{n \in \mathbb{N}}$ is bounded. This contradiction completes the proof.

We conclude with a simple application of the theorem established. Let D and φ be as in Example 1.5. Then, clearly, D is biholomorphically equivalent to a polydisc which, being a bounded analytic polyhedron, has property C. Since U_φ has purely continuous spectrum, it follows from Theorem 5.6 that the spectrum of U_φ is in fact purely absolutely continuous.

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