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ON BANACH ALGEBRAS WHICH ARE HILBERT SPACES

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Let F denote a field that is either the real field $\boldsymbol{R}$ or the complex field $\boldsymbol{C}$. Let H denote the real quaternion algebra.

An algebra A over F with identity e is called a *Hilbert algebra with identity* if A is a Hilbert space such that the norm derived from an inner product has the following properties:

1° $\|e\| = 1$,

2° $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ for $a, b \in A$.

If $F = \boldsymbol{R}$, A is called a *real Hilbert algebra*, and if $F = \boldsymbol{C}$, a *complex Hilbert algebra*.

L. Ingelstam proved in [2] the following

Theorem. *A real Hilbert algebra with identity is isomorphic to $\boldsymbol{R}$, $\boldsymbol{C}$ or H.*

In the present note we shall give a new simpler proof of this theorem.

We shall begin our considerations with the following

Lemma. *Let A be a real Hilbert algebra with identity e. Let Φ be a multiplicative linear functional on A. Then Φ is injective.*

Proof. In virtue of the Riesz theorem, Φ represents in the form

$$\Phi(x) = (x, a), \tag{1}$$

where $x, a \in A$. It is well known that $\|\Phi\| = 1$ for an arbitrary multiplicative linear functional Φ on a Banach algebra with identity. Hence and by (1) we obtain

$$\|a\| = 1. \tag{2}$$

By the Cauchy-Schwarz inequality and by 1°, (2) we have

$$1 = \Phi(e) = (e, a) \leq \|e\| \cdot \|a\| = 1. \tag{3}$$

In particular $(e, a) = \|e\| \cdot \|a\|$. But the equality in the Cauchy-Schwarz occurs exactly in the case when both a and e are linearly dependent. Therefore we can write $a = \lambda e$, $\lambda \in \boldsymbol{R}$. Using 1°, (3) we obtain

$$\lambda = \lambda(e, a) = (a, e) = 1,$$

so actually we have $a = e$ and

$$\Phi(x) = (x, e). \tag{4}$$

Now let us consider the following linear operator

$$T_a(x) = a \cdot x, \quad a, x \in A.$$

Using 2° and the fact that $T_a(e) = a$ we obtain

$$\|T_a\| = \|a\|. \tag{5}$$

Let T_a^* denote the adjoint operator. By the elementary fact that $\|T_a^*\| = \|T_a\|$ and by (5) we obtain

$$\|T_a^*\| = \|a\|. \tag{6}$$

Now by the Cauchy-Schwarz inequality and by 1°, (6) we have

$$\|a\|^2 = \big(T_a(e), a\big) = \big(e, T_a^*(a)\big) \le \|e\| \cdot \|T_a^*(a)\| \le \|T_a^*\| \cdot \|a\| = \|a\|^2. \tag{7}$$

Hence, as before, $T_a^*(a) = \lambda e$ for some $\lambda \in \boldsymbol{R}$, and by 1°, (7) we have

$$\lambda = \lambda(e,e) = \big(e, T_a^*(a)\big) = \|a\|^2.$$

Consequently

$$T_a^*(a) = \|a\|^2 \cdot e. \tag{8}$$

Now by (8) we obtain for $a, b \in A$

$$(a \cdot b, a) = \big(T_a(b), a\big) = \big(b, T_a^*(a)\big) = \|a\|^2 \cdot (b, e). \tag{9}$$

By (9), 1° we have for $a \in A$

$$\begin{aligned} \big((a+e)\cdot a, a+e\big) - (a^2, a) &= \|a+e\|^2 \cdot (a,e) - \|a\|^2 \cdot (a,e) \\ &= \big(1 + 2(a,e)\big) \cdot (a,e). \end{aligned} \tag{10}$$

But

$$\big((a+e)\cdot a, a+e\big) - (a^2, a) = \|a\|^2 + (a^2, e) + (a, e). \tag{11}$$

By (4) we have

$$(a^2, e) = \varPhi(a^2) = \big(\varPhi(a)\big)^2 = (a,e)^2. \tag{12}$$

Taking account of (10), (11), (12) we obtain $\|a\|^2 = |(a,e)| = |\varPhi(a)|$. Hence $\ker \varPhi = \{0\}$. The proof of the lemma is completed.

Now we shall prove the main theorem. Let A be a real Hilbert algebra with identity. In order to prove that A is isomorphic to $\boldsymbol{R}$, $\boldsymbol{C}$ or H it is sufficient to show that A is a real division algebra (cf. Theorem 7, Chapter I, §14 of F. F. Bonsall, J. Duncan [1]). Let x be an arbitrary non-zero element of A. Let us consider the smallest closed subalgebra A_0 with identity containing x. A_0 is a commutative real Hilbert algebra with identity. Suppose that x has no inverse in A_0. Then x belongs to a certain maximal ideal M. Obviously, M is closed. Let ϕ be the canonical homomorphism of A_0 onto A_0/M. A_0/M is a commutative real division algebra and therefore A_0/M is isomorphic to $\boldsymbol{R}$ or $\boldsymbol{C}$. In both cases A_0/M contains a closed subalgebra B isomorphic to $\boldsymbol{R}$. Let η be an isomorphism of B onto $\boldsymbol{R}$. Since ψ is continuous, it follows that $\psi^{-1}(B)$ is a real Hilbert algebra with identity. $\varPhi = \eta \circ \psi$ is a multiplicative linear functional on $\psi^{-1}(B)$. In virtue of the lemma, $\varPhi$ is injective. In particular, $M = \{0\}$ contrary to our assumption of the non-triviality of M. Consequently x has inverse in A_0, and so in A. A is then a real division algebra. This completes the proof.

An immediate consequence of our theorem is the following

Corollary. *A complex Hilbert algebra with identity is isomorphic to* $\boldsymbol{C}$.

Proof. Consider a complex Hilbert algebra A with identity as a real Hilbert algebra with the inner product $\langle x, y\rangle = \mathrm{Re}(x,y)$ for $x, y \in A$. In virtue of our theorem, A is isomorphic to $\boldsymbol{R}$, $\boldsymbol{C}$ or H. Among the last three algebras $\boldsymbol{C}$ is the only complex Banach algebra. This remark completes the proof. □

References

1. F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin–Heidelberg–New York, 1973.
2. L. Ingelstam, *Hilbert algebras with identity*, Bull. Amer. Math. Soc. **69** (1963), 794–796.