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The generalized spectral analysis of $i^{-1}\frac{d}{dx}+q$ with real almost periodic q, with applications to harmonic analysis

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RÉSUMÉ - *A l'opérateur* $A_q = i^{-1}\frac{d}{dx} + q$, *$q$ étant une fonction réelle continue presque périodique dans* $\mathbf{R}$, *on associe l'opérateur transformé de Fourier de l'opérateur analogue à* A_q *dans le compactifié de* BOHR *de* $\mathbf{R}$, *et on en effectue l'analyse spectrale. On prouve que la seule à un facteur numérique près fonction propre de l'opérateur initial, correspondant à la valeur propre zéro, est presque totalement ergodique. On donne des conditions nécessaires et suffisantes pour que cette fonction soit totalement ergodique.*

0. Introduction.

Let $AP(\mathbf{R})$ ($AP_{\mathbf{R}}(\mathbf{R})$) denote the space of all (real) continuous almost periodic functions on $\mathbf{R}$. Given $q \in AP_{\mathbf{R}}(\mathbf{R})$, define a first order ordinary differential operator A_q to be

$$A_q u = i^{-1} u' + qu$$

for all u in the Sobolev space $H_1(\mathbf{R})$. Regarded as a densely defined operator on $L^2(\mathbf{R})$, A_q is self-adjoint. Its spectrum is the whole real line inasmuch as for any $\mu \in \mathbf{R}$, $u_{q\mu}(x) = \exp\left(i \int_0^x (\mu - q(u))\, du \right)$ $(x \in \mathbf{R})$ is a generalized eigenfunction of A_q with eigenvalue μ.

With the usual notation $l^2(\mathbf{R})$ for the Hilbert space of all square summable complex sequences on $\mathbf{R}$, and with $\hat{q}(\mu)$ standing for the μth Fourier coefficient of q, i. e.

$$\hat{q}(\mu)=\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T} q(x)\exp(-i\mu x)\,dx \quad (\mu\in\mathbf{R}),$$

let $\mathscr{A}_q$ be defined on $D(\mathscr{A}_q)=\{(a_\mu)\in l^2(\mathbf{R}):\ (\mu a_\mu)\in l^2(\mathbf{R})\}$ as

$$(\mathscr{A}_q(a_\mu))_\nu=\nu a_\nu+\sum_{\mu\in\mathbf{R}}\hat{q}(\nu-\mu)a_\mu .$$

Being the sum of the unbounded self-adjoint operator $(a_\mu)\to(\mu a_\mu)$ and of the bounded self-adjoint operator $(a_\mu)\to(\sum_{\nu\in\mathbf{R}}\hat{q}(\mu-\nu)\,a_\nu)$, $\mathscr{A}_q$ is self-adjoint. That the second summand in the above sum is bounded is seen as follows: Let $\tilde{\mathbf{R}}$ be the Bohr compactification of $\mathbf{R}$ with probabilistic Haar measure $\mathbf{P}$, and let $\alpha\colon \mathbf{R}\to\tilde{\mathbf{R}}$ be the canonical homomorphism. Given $(a_\mu)\in l^2(\mathbf{R})$, denote by X the Fourier transform of (a_μ) defined in a unique manner as an element of $L^2(\tilde{\mathbf{R}})$. Let Q be the unique element of $C_{\mathbf{R}}(\tilde{\mathbf{R}})$ (the continuous real functions on $\tilde{\mathbf{R}}$) such that $Q(\alpha(t))=q(t)$ for all $t\in\mathbf{R}$. Then the inverse Fourier transform of QX is $(\sum_{\nu\in\mathbf{R}}\hat{q}(\mu-\nu)\,a_\nu)$, and so, by Plancherel's theorem,

$$\|(\sum_{\nu\in\mathbf{R}}\hat{q}(\mu-\nu)\,a_\nu)\|_2\le\|q\|_\infty\,\|(a_\mu)\|_2 .$$

As the last passage suggests, one may think of $\mathscr{A}_q$ as being the Fourier transform of the operator A_q «transferred» on the Bohr compactification $\tilde{\mathbf{R}}$. A natural question arises how A_q and $\mathscr{A}_q$ are related to one another; this is a special instance of a general problem emerging in the study of pseudodifferential operators with almost periodic symbols (cf. [3], [14]). A fundamental theorem of Šubin [13] ensures that the spectra of both these operators coincide. Deeper relationships between A_q and $\mathscr{A}_q$ will be revealed in the course of the present paper as we perform a detailed spectral analysis of $\mathscr{A}_q$, and subsequently found thereupon a study of harmonic properties of $u_q=u_{q0}$, involving Banach means, ergodicity and related notions.

1. Spectral analysis of $\mathscr{A}_q$.

1.0. *Introduction.*

There is a close connection between the subject matter of this section and the theory of invariant subspaces of $L^2(G)$, where G is a compact Abelian group with ordered dual (cf. [7]). From the standpoint of the latter theory, the subsequent development can be viewed as an examination of a particular cocycle on $\widetilde{\mathbf{R}}$. Correspondingly, a considerable portion of the treatment will come as no surprise to experts in the field. Hovewer, our exposition seems to carry the results about cocycles a little further. A new short argument is used to identify $\mathcal{A}_q$ with the Fourier tranform of the generator of the group associated with a corresponding cocycle. The construction of a non-trivial cocycle offered here, based on a careful analysis of $\mathcal{A}_q$, seems to clarify much of the existing constructions of a similar kind.

Generally, our exposition is subordinate to the discussion of $\mathcal{A}_q$ in its own right. Recently, there has been a great interest in the spectral analysis of the operators defined in various functional spaces on the Bohr compactification of $\mathbf{R}^n$, associated with pseudodifferential operators with almost periodic symbols. So far as the author is aware, A_q and the Schrödinger operator with a periodic potential are until now the only differential operators with almost periodic coefficients, whose associated operators on $L^2(\widetilde{\mathbf{R}}^n)$ allow a complete spectral description.

Anticipating the interest in the paper of experts in differential equations unfamiliar with special chapters of harmonic analysis, our style will be largely expository with a minimal background for most of the arguments.

1.1. *Spectral resolution of* $\mathcal{A}_q$.

On letting $T_s f$ denote the translate of f by s, put

$$Y_t(\omega) = \exp\left(i \int_0^t T_{\alpha(u)} Q(\omega)\, du \right)$$

for all $t \in \mathbf{R}$ and all $\omega \in \widetilde{\mathbf{R}}$.

Throughout we shall use systematically the language of probability theory. Accordingly, $(t, \omega) \to Y_t(\omega)$ will be viewed as a stochastic process carried by $(\widetilde{\mathbf{R}}, \mathbf{P})$.

As easily seen, $\{Y_t\}$ satisfies the following conditions:

(i) $\{Y_t\}$ is continuous as a mapping from $\mathbf{R}$ into $L^2(\widetilde{\mathbf{R}})$,

(ii) for every $t \in \mathbf{R}$, $|Y_t| = 1$ almost surely (almost unitarity),

(iii) the random fields $\{Y_{s+t}\}$ and $\{Y_s T_{\alpha(s)} Y_t\}$, with parameter set $\mathbf{R}\times\mathbf{R}$, are stochastically equivalent.

Usually, a process satisfying (i) - (iii) is called a cocycle on $\tilde{\mathbf{R}}$.

Let $\tilde{A}_q$ be the self-adjoint operator on $L^2(\tilde{\mathbf{R}})$ that is unitarily equivalent to $\mathscr{A}_q$ by the Fourier tranform. $\tilde{A}_q$ is the sum of the generator of the strongly continuous unitary group $\{T_{\alpha(t)}\}$ on $L^2(\tilde{\mathbf{R}})$ times i^{-1} (the quasimomentum operator), and the bounded self-adjoint operator of multiplication by Q. By the Trotter product formula (cf.[11], th.8.31), $i\tilde{A}_q$ is the generator of the strongly continuous unitary group $\{U_t\}$ defined as

$$U_t=\lim_{n\to\infty}\left(T_{\alpha\left(\frac{t}{n}\right)}\exp\left(i\frac{t}{n}Q\right)\right)^n,$$

the limit being taken in the strong operator topology (here $\exp\left(i\frac{t}{n}Q\right)$ stands for the corresponding multiplication operator). Apparently, for any $t\in\mathbf{R}$ and $n\in\mathbf{N}$

$$\left(T_{\alpha\left(\frac{t}{n}\right)}\exp\left(i\frac{t}{n}Q\right)\right)^n=\exp\left(i\frac{1}{n}\sum_{k=1}^{n}T_{\alpha\left(\frac{k}{n}t\right)}Q\right)T_{\alpha(t)}.$$

Moreover

$$\lim_{n\to\infty}\exp\left(i\frac{1}{n}\sum_{k=1}^{n}T_{\alpha\left(\frac{k}{n}t\right)}Q\right)=Y_t$$

in the sense of $L^2(\tilde{\mathbf{R}})$, the pointwise limit being evident and the equality next following on application of Lebesque's dominated convergence theorem. Thus

$$U_t=Y_t\,T_{\alpha(t)}. \tag{1.1}$$

Denote by F the projection-valued measure associated with $\{U_t\}$ or, equivalently, with $\tilde{A}_q$. With respect to F, $L^2(\tilde{\mathbf{R}})$ decomposes into the three mutually orthogonal subspaces: pure point, continuous singular and absolutely continuous. It is a well known fact about the projection-valued measures associated with cocycles that one of these subspaces must be all of $L^2(\tilde{\mathbf{R}})$ (cf. [7], th. 14, p. 25). Below we give a simple proof of this assertion. One point of the proof will be relevant to the subsequent development.

For any $\tau \in \mathbf{R}$, let χ_τ denote the continuous character of $\tilde{\mathbf{R}}$ such that $\chi_\tau(\alpha(t)) = \exp(i\tau t)$ for all $t \in \mathbf{R}$. Given $\tau \in \mathbf{R}$ and $f \in L^2(\tilde{\mathbf{R}})$, set

$$V_\tau f = \chi_\tau f.$$

Then $\{V_\tau\}$ is another unitary (not strongly continuous) group on $L^2(\tilde{\mathbf{R}})$.

As easily seen, the families $\{U_t\}$ and $\{V_\tau\}$ verify the Weyl commutation relation

$$U_t V_\tau = \exp(it\tau) V_\tau U_t .$$

Using this relation, one readily finds for any $\tau \in \mathbf{R}$ and any Borel subset B of $\mathbf{R}$

$$F(B) V_\tau = V_\tau F(B - \tau) \tag{1.2}$$

whence

$$(F(B) V_\tau f,\ V_\tau f) = (F(B-\tau) f, f)$$

whenever $f \in L^2(\tilde{\mathbf{R}})$.

By the latter formula, each of the three parts of $L^2(\tilde{\mathbf{R}})$ relative to F is invariant for $\{V_t\}$, and as such, by a theorem of Wiener (cf.[6], th. 7.10.1), consists of all the functions of $L^2(\tilde{\mathbf{R}})$ supported on some fixed Borel subset C of $\tilde{\mathbf{R}}$. Of course, each of these parts is also invariant for $\{U_t\}$, so given $t \in \mathbf{R}$, the function

$$U_t 1_C = Y_t T_{\alpha(t)} 1_C$$

vanishes off C (here 1_C denotes the characteristic function of C). Y_t being a unitary function, we have

$$\mathbf{P}\ ((C - \alpha(t)) \Delta C) = 0$$

for all $t \in \mathbf{R}$. Since the standard flow on $\tilde{\mathbf{R}}$:

$$\varphi_t(\omega) = \omega + \alpha(t) \quad (\omega \in \tilde{\mathbf{R}},\ t \in \mathbf{R})$$

is ergodic, it follows that either $\mathbf{P}(C) = 0$ or $\mathbf{P}(\tilde{\mathbf{R}} \backslash C) = 0$. The proof is complete.

By unitary equivalence, we obtain

THEOREM 1.1. *The spectrum of $\mathcal{A}_q$ is either pure point, purely continuous singular or purely absolutely continuous.*

1.2. *A function.*

Let φ_q be the function defined for any t in $\mathbf{R}$ to be the mean value of the almost periodic function $x \to \exp\left(i \int_x^{x+t} q(u)\, du\right)$, i. e.

$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} \exp\left(i \int_x^{x+t} q(u)\, du\right) dx.$$

Since, for $x,\ t \in \mathbf{R}$, we have

$$Y_t(\alpha(x)) = \exp\left(i \int_0^t q(x+u)\, du\right) = \exp\left(i \int_x^{x+t} q(u)\, du\right),$$

the function $\omega \to Y_t(\omega)$ is the continuous extension on $\tilde{\mathbf{R}}$ of $x \to \exp\left(i \int_x^{x+t} q(u)\, du\right)$. Therefore

$$\varphi_q(t) = \mathbf{E} Y_t \quad (t \in \mathbf{R}),$$

where $\mathbf{E}$ stands for the expectation operator relative to $\mathbf{P}$. On the other hand, by (1.1),

$$\mathbf{E} Y_t = (U_t 1, 1) \quad (t \in \mathbf{R})$$

is the Fourier transform of the measure $(F(-B)\,1, 1)$ (B a Borel subset of $\mathbf{R}$). The latter is of pure type, and its type coincides with the type of the spectrum of $\mathcal{A}_q$.

It thus appears that the determination of the type of the spectrum of $\mathcal{A}_q$ reduces to finding the type of the measure whose Fourier transform is φ_q.

As a simple result involving φ_q, we have the following.

THEOREM 1.2. *The spectrum of $\mathcal{A}_q$ is purely continuous or pure point according as the limit*

$$\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}|\varphi_q(x)|^2\,dx$$

does or does not vanish.

The theorem follows immediately from one due to Wiener (cf. [12], th. 5.6.9.), ensuring that

$$\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}|\varphi_q(x)|^2\,dx=\sum_{\mu\in\mathbf{R}}(F(\{\mu\})\,1,1)^2.$$

1.3. *Pure point spectrum.*

Suppose X is a unit eigenvector of $\widetilde{A}_q$ with eigenvalue λ_0. Then, as easily seen from (1.2), for every $\lambda\in\mathbf{R}$, $\chi_\lambda X$ is an eigenvector with eigenvalue $\lambda+\lambda_0$. Accordingly, for all $\lambda\neq 0$, we have

$$\mathbf{E}\,(|X|^2\,\overline{\chi_\lambda})=0,$$

and so $|X|$ is essentially constant, in fact essentially equal to one. Now the afore-mentioned theorem of Wiener on subpaces invariant for $\{V_\tau\}$ applies to the effect that $\{\chi_\lambda X\colon \lambda\in\mathbf{R}\}$ is linearly dense in $L^2(\widetilde{\mathbf{R}})$. Consequently, $\{\chi_\lambda X\colon \lambda\in\mathbf{R}\}$ is a complete orthonormal set, and, by unitary equivalence, we get the following.

THEOREM 1.3. *If $\mathcal{A}_q$ has pure point spectrum, then every real number is a simple eigenvalue of $\mathcal{A}_q$.*

Note that the above argument also shows that if $\mathcal{A}_q$ has pure point spectrum, then there exists an almost unitary function X in ker $\widetilde{A}_q$. Since $\{U_t\}$ leaves ker $\widetilde{A}_q$ unaffected, we have for all $t\in\mathbf{R}$

$$Y_t\,T_{\alpha(t)}\,X=X$$

whence

$$Y_t=XT_{\alpha(t)}\,\overline{X}.$$

Thus $\{Y_t\}$ takes the form of a so-called coboundary. According to a general definition, a cocycle $\{Z_t\}$ is a coboundary if there exists an almost unitary random variable V such that the processes $\{Z_t\}$ and $\{VT_{\alpha(t)}\,\overline{V}\}$ are stochastically equivalent. Hereafter, in agreement with the terminology employed in [9], such a V will be called an invariant section of $\{Z_t\}$.

1.4. *q-groups.*

Denote by M_q the frequency module of q, i.e. the subgroup of $\mathbf{R}$ generated by $\{\mu \in \mathbf{R}: \hat{q}(\mu) \neq 0\}$.

Any compact Abelian group G whose dual $\hat{G}$ is a discrete subgroup of $\mathbf{R}$ containing M_q will be called a q-group.

Suppose G is a q-group. Then there exist a continuous homomorphism α_G from $\mathbf{R}$ onto a dense subgroup of G, and Q^G in $C_{\mathbf{R}}(G)$ with $Q^G(\alpha_G(t)) = q(t)$ for all $t \in \mathbf{R}$. Putting

$$\{Y_t^G\} = \left\{ \exp\left(i \int_0^t T_{\alpha_G(u)} Q^G \, du \right) \right\}$$

defines a cocycle on G. The latter gives rise to the strongly continuous unitary group $\{U_t^G\}$ on $L^2(G)$ defined by a formula analogous to (1.1). A moment's reflection shows that the generator of $\{U_t^G\}$, $\tilde{A}_q^G$, is, by the Fourier tranform, unitarily equivalent to the operator $\mathcal{A}_q$ restricted to $l^2(\hat{G})$, $l^2(\hat{G})$ being embedded in an obvious manner in $l^2(\mathbf{R})$.

All of our previous work can be repeated for G in place of $\tilde{\mathbf{R}}$. The only difference is that in the case where $\mathcal{A}_{q|l^2(\hat{G})}$ has pure point spectrum the eigenvalues of $\mathcal{A}_{q|l^2(\hat{G})}$ range over a coset of $\hat{G}$ in $\mathbf{R}$ and there may happen that the kernel of $\mathcal{A}_{q|l^2(\hat{G})}$ is void. In this case $\{Y_t^G\}$ is merely a trivial cocycle, which means, by definition, that $\{Y_t^G\}$ is stochastically equivalent to the process $\{\exp(i\lambda t) X T_{\alpha_G(t)} \overline{X}\}$, where $\lambda \in \mathbf{R}$ and X is an almost unitary random variable on the probability space $(G, \mathbf{P}_G)$ with $\mathbf{P}_G$ being the Haar measure on G. That $\{Y_t^G\}$ is trivial follows immediately upon taking λ to be any eigenvalue of $\tilde{A}_G$ and X a corresponding unit eigenvector.

An easy argument based on the purity of the spectra shows that the spectra of $\mathcal{A}_q$ and $\mathcal{A}_{q|l^2(\hat{G})}$ are of that same type.

On account of the above remarks and the last result of the preceding section the first part of our next theorem is clear.

THEOREM 1.4. *If $\mathcal{A}_q$ has pure point spectrum, then for every q-group G, the cocycle $\{Y_t^G\}$ is trivial, and in the case $G = \tilde{\mathbf{R}}$, it is a coboundary. Conversely, if for some q-group G, the cocycle $\{Y_t^G\}$ is trivial, then $\mathcal{A}_q$ has pure point spectrum.*

To prove the second part, note that for any $t \in \mathbf{R}$

$$\varphi_q(t) = \mathbf{E}_G Y_t^G = \exp(i\lambda t)\, \mathbf{E}_G (X T_{\alpha_G(t)} \overline{X}) \tag{1.3}$$

$$= \sum_{\gamma \in \hat{G}} |\mathcal{F}_G X(\gamma)|^2 \exp(i(\lambda - \gamma) t),$$

where $\mathcal{F}_G$ denotes the Fourier transform from $L^2(G)$ onto $l^2(\hat{G})$, and $\mathbf{E}_G$ denotes the expectation operator relative to $\mathbf{P}_G$. Clearly, φ_q is the Fourier transform of a purely atomic measure and so the theorem follows.

1.5. *A representation theorem.*

It follows from what precedes that if $\mathcal{A}_q$ has pure point spectrum, then the cocycle $\{Y_t^{\hat{M}_q}\}$ is trivial and, in accordance with (1.3), for any unit eigenvector X of $\widetilde{A}_q^{\hat{M}_q}$ with eigenvalue λ, the identity

$$\varphi_q(t) = \sum_{\gamma \in M_q} |\mathcal{F}_{\hat{M}_q} X(\gamma)|^2 \exp(i(\lambda - \gamma) t)$$

holds for all $t \in \mathbf{R}$. Taking into account that $\mathcal{F}_{\hat{M}_q} X$ is a unit eigenvector of $\mathcal{A}_{q|l^2(M_q)}$ with eigenvalue λ, we arrive at the following.

THEOREM 1.5. *If $\mathcal{A}_q$ has pure point spectrum, then for any unit eigenvector (a_μ) of $\mathcal{A}_{q|l^2(M_q)}$ with eigenvalue λ and for all $t \in \mathbf{R}$, one has*

$$\varphi_q(t) = \sum_{\mu \in M_q} |a_\mu|^2 \exp(i(\lambda - \mu) t).$$

1.6. *Pure point spectrum (continued).*

In this section, we show that the class of all q in $AP_{\mathbf{R}}(\mathbf{R})$ such that $\mathcal{A}_q$ has pure point spectrum is non-void and in fact splits into two disjoint subclasses, the decomposition being of direct relevance in the next chapter.

One of these subclasses is made up of all those q in $AP_{\mathbf{R}}(\mathbf{R})$ for which u_q is almost periodic (recall that $u_q(x) = \exp\left(-i\int_0^x q(u)\, du\right)$ $(x \in \mathbf{R})$); in view of the argument theorem of Bohr (cf. [4], [8]), this last condition may be replaced by the condition that the function

$x \to \int_0^x q(u)\,du - \hat{q}(0)\,x$ be almost periodic. It is easily verified that, given $q \in AP_{\mathbf{R}}(\mathbf{R})$, if u_q belongs to $AP_{\mathbf{R}}(\mathbf{R})$, then the continuous extension of u_q on $\tilde{\mathbf{R}}$ is an invariant section of $\{Y_t\}$, and so, by theorem 1.4, $\mathscr{A}_q$ has pure point spectrum.

In view of theorem 1.2, the other subclass has to consist of all those q in $AP_{\mathbf{R}}(\mathbf{R})$ for which

$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |\varphi_q(x)|^2\,dx > 0$$

without u_q being almost periodic. By modifying a well-known Bohr construction, we shall show that this subclass is non-void.

Let λ_1 and λ_2 be two rationally independent real numbers. Let n_1 and n_2 be two functions carrying $\mathbf{N}$ into $\mathbf{Z}$ such that for any $k \in \mathbf{N}$, $a_k = \lambda_1 n_1(k) + \lambda_2 n_2(k)$ satisfies $k^{-2/3} \le a_k \le 2k^{-2/3}$. Put

$$q(x) = \sum_{k=1}^{\infty} a_k^2 \sin a_k x \quad (x \in \mathbf{R}).$$

Clearly $\hat{q}(0) = 0$. Moreover

$$\int_0^x q(u)\,du = \sum_{k=1}^{\infty} a_k (1 - \cos a_k x) = \sum_{k=1}^{\infty} 2a_k \sin^2(a_k x/2).$$

Bearing in mind that $|\sin t| \ge |t|/2$ for $|t| \le 1$, we may write whenever $|x| \ge 1$

$$\int_0^x q(u)\,du \ge 8^{-1} x^2 \sum_{k \ge |x|^{3/2}} k^{-2} \ge 8^{-1} x^2 \int_{|x|^{3/2}-1}^{\infty} u^{-2}\,du = 8^{-1}(|x|^{3/2}-1)^{-1} x^2$$

whence

$$\lim_{|x|\to\infty} \int_0^x q(u)\,du = +\infty.$$

Thus, by the argument theorem of Bohr, u_q is not almost periodic.

Take $\mathbf{R}^2/(2\pi\mathbf{Z})^2$ for a q-group with $\alpha_G(t) = [(\lambda_1 t, \lambda_2 t)]$ for all $t \in \mathbf{R}$ and

$$Q^G([(\Theta_1, \Theta_2)]) = \sum_{k=1}^{\infty} a_k^2 \sin(n_1(k)\Theta_1 + n_2(k)\Theta_2)$$

for all $[(\Theta_1, \Theta_2)] \in G$. Since $\sum_{k=1}^{\infty} a_k^2 < +\infty$, there exists Y in $L^2(G)$ such that

$$Y([(\Theta_1, \Theta_2)]) \sim \sum_{k=1}^{\infty} a_k \cos(n_1(k)\Theta_1 + n_2(k)\Theta_2).$$

Put $X = \exp(iY)$. One easily verifies that X is an invariant section of $\{Y_t^G\}$. Thus, in view of theorem 1.4, $\mathscr{A}_q$ has pure point spectrum.

1.7. *Purely continuous spectrum.*

The aim of this section is to prove the following.

THEOREM 1.6. *Let (α_k) be a sequence of rationally independent real numbers such that $\sum_{k=1}^{\infty} |\alpha_k| < +\infty$. Let p be a non-zero real continuous periodic function on* **R** *with mean value zero. Let*

$$q(x) = \sum_{k=1}^{\infty} \alpha_k\, p(\alpha_k x) \quad (x \in \mathbf{R}).$$

Then $\mathscr{A}_q$ has purely continuous spectrum.

PROOF. We may suppose that p has period 2π.

Since the spectrum of $x \longrightarrow \exp\left(i\int_x^{x+t} \alpha_k\, p(\alpha_k u)\, du\right)$ $(t \in \mathbf{R},\ k \in \mathbf{N})$ is contained in $\alpha_k \mathbf{Z}$, the functions $x \longrightarrow \exp\left(i\int_x^{x+t} \alpha_k\, p(\alpha_k u)\, du\right)$, $k = 1, 2, \ldots$, are independent (more correctly, their continuous extensions on $(\tilde{\mathbf{R}}, \mathbf{P})$ are so). Thus, for any $n \in \mathbf{N}$ and $t \in \mathbf{R}$, we have

$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} \prod_{k=1}^{n} \exp\left(i\int_x^{x+t} \alpha_k\, p(\alpha_k u)\, du\right) dx$$

$$= \prod_{k=1}^{n} \lim_{T\to\infty} (2T)^{-1} \int_{T}^{T} \exp\left(i\int_x^{x+t} \alpha_k\, p(\alpha_k u)\, du\right) dx$$

$$= \prod_{k=1}^{n} \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} \exp\left(i \int_{x}^{x+\alpha_k t} p(u)\, du\right) dx$$

whence

(1.4)
$$\varphi_q(t) = \lim_{n\to\infty} \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} \prod_{k=1}^{n} \exp\left(i \int_{x}^{x+t} \alpha_k\, p(\alpha_k u)\, du\right) dx$$
$$= \prod_{k=1}^{\infty} \varphi_p(\alpha_k t).$$

Since $\hat{p}(0)=0$, φ_p is periodic (with period 2π), and furthermore the functions $t \to \varphi_p(\alpha_k t)$, $k=1, 2, \dots$, are independent. Consequently, for any $n \in \mathbf{N}$, we have

(1.5)
$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |\varphi_q(t)|^2\, dt \leq \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} \prod_{k=1}^{n} |\varphi_p(\alpha_k t)|^2\, dt$$
$$= \prod_{k=1}^{n} \lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |\varphi_p(\alpha_k t)|^2\, dt = \left(\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |\varphi_p(t)|^2\, dt\right)^n.$$

On the other hand, since the function $x \to \int_0^{\infty} p(u)\, du$ is periodic (with period 2π), $\mathscr{A}_p$ has pure point spectrum. By theorem 1.5 and the periodicity of φ_p, $\ker \mathscr{A}_{p|l^2(\mathbf{Z})}$ is non-void, and if (a_μ) is any unit vector in $\ker \mathscr{A}_{p|l^2(\mathbf{Z})}$, then for all $t \in \mathbf{R}$

$$\varphi_p(t) = \sum_{\mu\in\mathbf{Z}} |a_\mu|^2 \exp(-i\mu t).$$

Of course, $\sum_{\mu\in\mathbf{Z}} |a_\mu|^4 < 1$ unless $a_\mu = 0$ for all but one μ, say μ'. The latter case is impossible for otherwise $1_{\{\mu'\}}$ would belong to $\ker \mathscr{A}_{p|l^2(\mathbf{Z})}$ and consequently, for all integers $\mu \neq 0$, we would have $\hat{p}(\mu)=0$. This jointly with the assumption that $\hat{p}(0)=0$ would imply $p=0$, a contradiction. Thus

(1.6)
$$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |\varphi_p(t)|^2\, dt = \sum_{\mu\in\mathbf{Z}} |a_\mu|^4 < 1,$$

and on letting n tend to infinity in (1.5), we get

$$\lim_{T\to\infty} (2T)^{-1}\int_{-T}^{T} |\varphi_q(t)|^2\, dt=0.$$

In view of theorem 1.2, this establishes that $\mathscr{A}_q$ has purely continuous spectrum.

1.8. *Refinements.*

By a suitable choice of (α_k) in the preceding construction, either of the two kinds of the continuous spectrum of $\mathscr{A}_q$ can be obtained.

First we reveal a choice of (α_k) so that $\mathscr{A}_q$ has purely continuous singular spectrum.

For all $n\in\mathbf{N}$, let β_n be a positive number with the property that

$$|\exp(ix)-1|<\beta_n \quad (x\in\mathbf{R})$$

implies

$$|\varphi_p(x)-1|<(n+1)^{-2}.$$

Pick α_1 and x_1 in $\mathbf{R}$ so that $|\alpha_1|\leq 1$, $|x_1|\geq 1$ and

$$|\exp(i\alpha_1 x_1)-1|<\beta_1 .$$

Suppose we have chosen $\alpha_1, \dots, \alpha_n$ and $x_1, \dots, x_n$ in $\mathbf{R}$ so that $|\alpha_k|\leq k^{-2}$, $|x_k|\geq k$ $(k=1, \dots, n)$, $\{\alpha_1, \dots, \alpha_n\}$ is a rationally independent set, and

$$|\exp(i\alpha_k x_l)-1|<\beta_k \quad (1\leq k, l\leq n).$$

In the $(n+1)$-*th* step, we select α_{n+1} rationally independent of $\alpha_1, \dots \alpha_n$ so small that $|\alpha_{n+1}|\leq(n+1)^{-2}$ and

$$|\exp(i\alpha_{n+1} x_l)-1|<\beta_{n+1} \quad (l=1, \dots, n).$$

Next, by independency, we find x_{n+1} in $\mathbf{R}$ such that $|x_{n+1}|\geq n+1$ and

$$|\exp(i\alpha_k x_{n+1})-1|<\beta_k \quad (k=1, \dots, n+1).$$

Continuing the process, we obtain two infinite sequences (α_k) and (x_k) such that the set $\{\alpha_k\colon k\in\mathbf{N}\}$ is rationally independent, $|\alpha_k|\leq k^{-2}$ and $|x_k|\geq k$ $(k\in\mathbf{N})$, and

$$|\exp(i\alpha_k x_l)-1|<\beta_k \quad (k, l\in\mathbf{N}).$$

Letting q be as in the preceding section, by (1.4), we obtain for all $l\in\mathbf{N}$

$$|\varphi_q(x_l)|\geq \prod_{k=1}^{\infty}(1-(k+1)^{-2})>0$$

whence

$$\overline{\lim_{|x|\to\infty}}\, |\varphi_q(x)|>0.$$

Consequently, φ_q cannot be the Fourier transform of an integrable function on $\mathbf{R}$, and so $\mathscr{A}_q$ must have purely continuous singular spectrum.

Now, on utilizing an argument due to C. C. Moore, we find (α_k) such that $\mathscr{A}_q$ has purely absolutely continuous spectrum.

On account of (1.6), there exists $I=(-a,-b)\cup(c,d)$ with $a,b,c,d>0$ such that $|\varphi_p(x)|\leq\delta<1$ for $x\in I$. Pick $\alpha_k\,(k\in\mathbf{N})$ so that $\alpha_k=k^{-2}+o(k^{-2})$ and $\{\alpha_k\colon k\in\mathbf{N}\}$ is a rationally independent set. Letting $c(x)$ equal the cardinality of $\{k\in\mathbf{N}\colon \alpha_k x\in I\}$, we see that $\underline{\lim}_{|x|\to\infty} c(x)|x|^{-1/2}>0$. For q as in the preceeding section, we find by (1.4)

$$|\varphi_q(x)|\leq\delta^{c(x)}\leq A\exp(-B|x|^{1/2})$$

with some $A,B>0$. Thus $\varphi_q\in L^1(\mathbf{R})$, and so, by a theorem from harmonic analysis (cf. [12], th. 1.5.1), φ_q is the Fourier transform of an integrable function on $\mathbf{R}$. Correspondingly, $\mathscr{A}_q$ has purely absolutely continuous spectrum.

2. **Harmonic properties of u_q.**

2.1. *Fourier expansion of u_q.*

We start with a brief review of some notions and facts from the harmonic analysis of bounded functions.

A linear continuous functional m on $L^\infty(\mathbf{R})$ is called a Banach mean on $L^\infty(\mathbf{R})$ if it satisfies the following conditions:

(i) $m(1)=1=\|m\|$,

(ii) $m(T_s f)=m(f)$ for every $f\in L^\infty(\mathbf{R})$ and every $s\in\mathbf{R}$.

A celebrated theorem of Banach [2] ensures the existence of at least one Banach mean on $L^\infty(\mathbf{R})$.

If m is Banach mean on $L^\infty(\mathbf{R})$, then under the scalar product derived from the map $L^\infty(\mathbf{R})\times L^\infty(\mathbf{R})\ni(f,g)\to m(f\bar{g})\in\mathbf{C}$, $L^\infty(\mathbf{R})/ /\{f\in L^\infty(\mathbf{R})\colon\ m(|f|^2)=0\}$ is a pre-Hilbert space. The functions

$x \to \exp(i\mu x)$ $(\mu \in \mathbf{R})$ generate an orthonormal set in this space. Given $f \in L^\infty(\mathbf{R})$ and $\mu \in \mathbf{R}$, the Fourier coefficient of the class of f with respect to the class of $x \to \exp(i\mu x)$, $\hat{f}^m(\mu)$, is called the μth Fourier coefficient of f with respect to m. Of course, $\hat{f}^m(\mu) = m_x(f(x) \exp(-i\mu x))$, where the subscript x indicates that the averaging process refers to the variable x. By Bessel's inequality, one has

$$\sum_{\mu \in \mathbf{R}} |\hat{f}^m(\mu)|^2 \leq m(|f|^2).$$

Applied to u_q, the above inequality shows that, with μ running over $\mathbf{R}$, the sequence $(\hat{u}_q{}^m(\mu))$ is in $l^2(\mathbf{R})$ and has norm ≤ 1. Actually much more is true of $(\hat{u}_q{}^m(\mu))$, as shows the following.

THEOREM 2.1. *If m is a Banach mean on $L^\infty(\mathbf{R})$, then the sequence $(\hat{u}_q{}^m(\mu))$ is in ker $\mathcal{A}_q$.*

PROOF. The starting point is the identity

$$\mu u_{q(-\mu)} + q u_{q(-\mu)} = i u_{q(-\mu)}' \tag{2.1}$$

valid for all $\mu \in \mathbf{R}$. Firsts, it implies that given $\mu \in \mathbf{R}$, the function $u_{q(-\mu)}'$ is uniformly continuous and as such may be uniformly approximated by function-minus-translate difference quotients. Since any Banach mean vanishes on such difference quotients, given a Banach mean m on $L^\infty(\mathbf{R})$, we have

$$m(u_{q(-\mu)}') = 0$$

whence by (2.1)

$$\mu \hat{u}_q{}^m(\mu) + m(q u_{q(-\mu)}) = 0. \tag{2.2}$$

Let (p_k) be a sequence of trigonometric polynomials uniformly approximating q. Since

$$m(p_k u_{q(-\mu)}) = \sum_{\nu \in \mathbf{R}} \hat{p}_k(\mu - \nu)\, \hat{u}_q{}^m(\nu),$$

we clearly have

$$m(q u_{q(-\mu)}) = \lim_{k \to \infty} \sum_{\nu \in \mathbf{R}} \hat{p}_k(\mu - \nu)\, \hat{u}_q{}^m(\nu). \tag{2.3}$$

On the other hand, the series $\sum_{\nu\in\mathbf{R}} \hat{q}(\mu-\nu)\,\hat{u}_q^m(\nu)$ converges and its sum equals $\lim_{k\to\infty} \sum_{\nu\in\mathbf{R}} \hat{p}_k(\mu-\nu)\,\hat{u}_q^m(\nu)$ as it is seen from the estimates

$$\sum_{\nu\in\mathbf{R}} |\hat{q}(\mu-\nu)\,\hat{u}_q^m(\nu)-\hat{p}_k(\mu-\nu)\,\hat{u}_q^m(\nu)|$$

$$\leq (\sum_{\nu\in\mathbf{R}} |\hat{q}(\nu)-\hat{p}_k(\nu)|^2)^{1/2} (\sum_{\nu\in\mathbf{R}} |\hat{u}_q^m(\nu)|^2)^{1/2} \leq ||q-p_k||_\infty.$$

This jointly with (2.2) and (2.3) yields

$$\mu\hat{u}_q(\mu)+\sum_{\nu\in\mathbf{R}} \hat{q}(\mu-\nu)\,\hat{u}_q^m(\nu)=0.$$

Noting that the last equality guarantees that $\sum_{\mu\in\mathbf{R}} \mu^2 |\hat{u}_q^m(\mu)|^2<+\infty$ ends the proof.

In particular, if the Fourier coefficients of u_q with respect to some Banach mean on $L^\infty(\mathbf{R})$ are not all zero, then $\mathcal{A}_q$ has pure point spectrum. Conversely, we have the following.

THEOREM 2.2. *If $\mathcal{A}_q$ has pure point spectrum, then there exists a Banach mean m on $L^\infty(\mathbf{R})$ such that the sequence $(\hat{u}_q^m(\mu))$ is non-zero.*

PROOF. Let X be an invariant section of $\{Y_t\}$. Denote by Φ the smallest translation-invariant subspace of $L^\infty(\mathbf{R})$ containing all of the functions $u_{q\mu}$ $(\mu\in\mathbf{R})$. With

$$\phi=\sum_{i=1}^{n} a_i T_{t_i} u_{q\mu_i} \quad (a_i\in\mathbf{C},\ t_i,\ \mu_i\in\mathbf{R})$$

in Φ we associate

$$n(\phi)=\sum_{i=1}^{n} a_i \exp(i\mu_i t_i)\, \mathbf{E}(\chi_{\mu_i} T_{\alpha(t_i)} X).$$

Since for all x in $\mathbf{R}$

$$|\phi(x)|=\left|\phi(x)\exp\left(i\int_0^x q(u)\,du\right)\right|$$

$$=\left|\sum_{i=1}^{n} a_i \exp(i\mu_i t_i) \exp(i\mu_i x) \exp\left(-i\int_0^{t_i} q(u+x)\,du\right)\right|$$

$$=\left|\sum_{i=1}^{n} a_i \exp(i\mu_i t_i)\, \chi_{\mu_i}(\alpha(x))\, \overline{Y}_{t_i}(\alpha(x))\right|,$$

it follows that

$$\|\phi\|_\infty=\left\|\sum_{i=1}^{n} a_i \exp(i\mu_i t_i)\, \chi_{\mu_i} \overline{Y}_{t_i}\right\|_\infty$$

$$=\left\|\sum_{i=1}^{n} a_i \exp(i\mu_i t_i)\, \chi_{\mu_i}\, T_{\alpha(t_i)} X\right\|_\infty.$$

This implies that the linear functional $\Phi \ni \phi \rightarrow n(\phi) \in \mathbf{C}$ is well-defined and has norm ≤ 1.

Since, for every $\mu \in \mathbf{R}$, the process

$$\{\exp(i\mu t)\chi_\mu \; T_{\alpha(t)} X\}=\{T_{\alpha(t)} \; (\chi_\mu X)\}$$

is strictly stationary, n is translation-invariant: $n(T_s\phi)=n(\phi)$ holds for all $s \in \mathbf{R}$. In virtue of an invariant prolongation theorem due to AGNEW and MORSE [1] (cf. also [5], th. 3.3.1), n can be extended to a translation-invariant functional on $L^\infty(\mathbf{R})$, the norm being unchanged. Continuing to denote the extended functional by n, consider on $L_{\mathbf{R}}^\infty(\mathbf{R})$ its real and imaginary parts n_1 and n_2, respectively. Of course, both these functionals are translation-invariant. Let n_j^i $(i, j=1,2)$ be the continuous positive linear functionals on $L_{\mathbf{R}}^\infty(\mathbf{R})$ defined by

$n_j^1(f)=\sup\{|n_j(g)|: |g|\leq f, g \in L_{\mathbf{R}}^\infty(\mathbf{R})\}$ for $f \geq 0$ in $L_{\mathbf{R}}^\infty(\mathbf{R})$, $n_j^2=n_j^1-n_j$

$(j=1, 2)$. Apparently, these functionals are also translation-invariant. On extending each of then canonically on $L^\infty(\mathbf{R})$, we find that

$$\mathcal{F}^{-1} X(\mu)=n(u_{q(-\mu)})=n_1^1(u_{q(-\mu)})-n_1^2(u_{q(-\mu)})$$

$$+in_2^1(u_{q(-\mu)})-in_2^2(u_{q(-\mu)})$$

for all $\mu \in \mathbf{R}$, and so at least one of the sequences $(n_j^i(u_{q(-\mu)}))$ $(i, j=1, 2)$, with μ running over $\mathbf{R}$, say $(n_1^1(u_{q(-\mu)}))$, is non-zero. Setting $m=n_1^1/\|n_1^1\|$, we see that m is a Banach mean on $L^\infty(\mathbf{R})$ and $(\hat{u}_q^m(\mu))$ is non-zero.

The proof is complete.

2.2. *Ergodic properties of* u_q.

The theorems established in the preceding section will be now used for discussing ergodic properties of u_q. Before proceeding further, we wish to recall some relevant notions and facts, and introduce one usefull concept.

An element f in $L^\infty(\mathbf{R})$ is called ergodic if it takes the same value on any Banach mean on $L^\infty(\mathbf{R})$. Alternatively, f is ergodic if and only if the closed convex envelope of $\{T_s f\colon s \in \mathbf{R}\}$ contains a constant function (cf. [10]). Such a constant function, if it exists, is the unique mean value of f. An element f in $L^\infty(\mathbf{R})$ is said to be totally ergodic if for every $\mu \in \mathbf{R}$, the function $f_\mu(x) = f(x)\exp(i\mu x)$ $(x \in \mathbf{R})$ is ergodic. As known, all almost periodic functions on $\mathbf{R}$ are totally ergodic.

For needs of the present paper, we introduce the notion of almost total ergodicity. An element f in $L^\infty(\mathbf{R})$ will be called almost totally ergodic if given two Banach means m and n on $L^\infty(\mathbf{R})$, the sequences $(\hat{f}^m(\mu))$ and $(\hat{f}^n(\mu))$ are linearly dependent. The qualification «almost totally ergodic» can be justified as follows. Suppose $f \in L^\infty(\mathbf{R})$ is almost totally ergodic. Then there is a Banach mean m on $L^\infty(\mathbf{R})$, such that if n is another Banach mean on $L^\infty(\mathbf{R})$, then $\hat{f}^n(\mu) = \lambda \hat{f}^m(\mu)$ for all $\mu \in \mathbf{R}$ with some $\lambda \in \mathbf{C}$. Since $\hat{f}^m(\mu) = 0$ for all but countably many μ's, the same is true with n in place of m. Accordingly, all but countably many f_μ's are ergodic with mean value zero.

By virtue of theorem 1.3 and theorem 2.1, u_q is almost totally ergodic.

If u_q is almost periodic, which happens, as we know, exactly when $x \to \int_0^x q(u)\,du - \hat{q}(0)\,x$ is almost periodic, then u_q is totally ergodic. On the other hand, if $\mathcal{A}_q$ has purely continuous spectrum or, equivalently, if $\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}|\varphi_q(x)|^2\,dx = 0$, then, by theorem 2.1, $(\hat{u}_q{}^m(\mu)) = 0$ for any Banach mean m on $L^\infty(\mathbf{R})$, and consequently u_q is totally ergodic also in this case. Actually, these are the only cases that u_q is totally ergodic. We have the following.

THEOREM 2.3 *Being almost totally ergodic,* u_q *is totally ergodic if and only if either* $x \to \int_0^x q(u)\,du - \hat{q}(0)\,x$ *is almost periodic or*

$\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |\varphi_q(x)|^2\, dx = 0$. *The latter case occurs exactly when all the Fourier coefficients of u_q vanish for any Banach mean on $L^\infty(\mathbf{R})$.*

PROOF. On account of the preceding remarks, we are left with proving that, first, if $(\hat{u}_q^m(\mu)) = 0$ for any Banach mean m on $L^\infty(\mathbf{R})$, then $\mathcal{A}_q$ has purely continuous spectrum; secondly, if u_q is totally ergodic and $\hat{u}_q^m(\mu) \neq 0$ for some $\mu \in \mathbf{R}$, m being a Banach mean on $L^\infty(\mathbf{R})$, then u_q is almost periodic.

The first paragraph follows immediately from theorem 2.2.

To prove the second, let $\varepsilon > 0$ be given. By the ergodicity of $u_{q(-\mu)}$, there exist positive numbers a_i $(i=1, \dots, n)$ with $\sum_{i=1}^{n} a_i = 1$ and real numbers t_i $(i=1, \dots, n)$ such that

$$\left\| \sum_{i=1}^{n} a_i T_{t_i} u_{q(-\mu)} - \hat{u}_q^m(\mu) \right\|_\infty < \varepsilon.$$

Since the expression on the left side is equal to

$$\left\| \sum_{i=1}^{n} a_i u_q T_{t_i} \overline{u}_{q(-\mu)} - \overline{\hat{u}_q^m(\mu)}\, u_q \right\|_\infty,$$

taking into account that

$$x \to (u_q T_{t_i} \overline{u}_{q(-\mu)})(x) = \exp\left(i \left(\int_x^{x+t_i} q(u)\, du + \mu (x + t_i) \right) \right) \qquad (i=1, \dots, n)$$

is almost periodic, we infer that $\overline{\hat{u}_q^m(\mu)}\, u_q$ is the uniform limit of almost periodic functions, and further that u_q is almost periodic.

The proof is complete.

We close the paper by noting that the above theorem exhibits the following curious phenomenon: if q is in the «non-almost periodic» subclass of the class of all $p \in AP_{\mathbf{R}}(\mathbf{R})$ such that $\mathcal{A}_p$ has pure spectrum, i.e. if $\lim_{T\to\infty} (2T)^{-1} \int_{-T}^{T} |\varphi_q(x)|^2\, dx > 0$ while $x \to \int_0^x q(u)\, du - \hat{q}(0)\, x$ is not almost periodic, then u_q is almost totally ergodic without being totally ergodic.

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