The equivalence of two definitions of compatible homography matrices

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Abstract

In many computer vision applications, one acquires images of planar surfaces from two different vantage points. One can use a projective transformation to map pixel coordinates associated with a particular planar surface from one image to another. The transformation, called a homography, can be represented by a unique, to within a scale factor, 3×3 matrix. One requires a different homography matrix, scale differences apart, for each planar surface whose two images one wants to relate. However, a collection of homography matrices forms a valid set only if the matrices satisfy consistency constraints implied by the rigidity of the motion and the scene. We explore what it means for a set of homography matrices to be compatible and show that two seemingly disparate definitions are in fact equivalent. Our insight lays the theoretical foundations upon which the derivation of various sets of homography consistency constraints can proceed.

Keywords: multiple homographies, homography matrix, fundamental matrix, latent variable

1. Introduction

Estimating the homography induced by a plane between two views from image measurements is a key step towards performing such tasks as metric rectification [10, 21], panorama generation [3], motion estimation [13, 17, 25], or camera calibration [35]. A closely related estimation issue is that of estimating *multiple* homographies between two views (see Figure 1). Successful tackling of this issue underpins many practical applications including non-rigid motion detection [19, 34], enhanced image warping [12], multiview 3D reconstruction [20], augmented reality [27], indoor navigation [28], multi-camera calibration [32], camera-projector calibration [26], or ground-plane recognition for object detection and tracking [1]. The estimation of a set of matrices representing homographies between two views cannot be simply reduced to the estimation of individual matrices. The reason for this is that homographies between two views are intrinsically interdependent and the corresponding matrices are subject to *compatibility constraints*. The identification of a full set of compatibility constraints is a challenging problem and its solution is predicated on finding a characterisation of interdependent homography matrices. As it turns out, multiple homography matrices can be characterised by adopting two different approaches. One of these exploits latent variables that capture the two underlying views and the planes giving rise to the homographies. The other is based on the correlation that exists between the fundamental matrix and the homographies between two views. The aim of this paper is to show that these two approaches are equivalent.

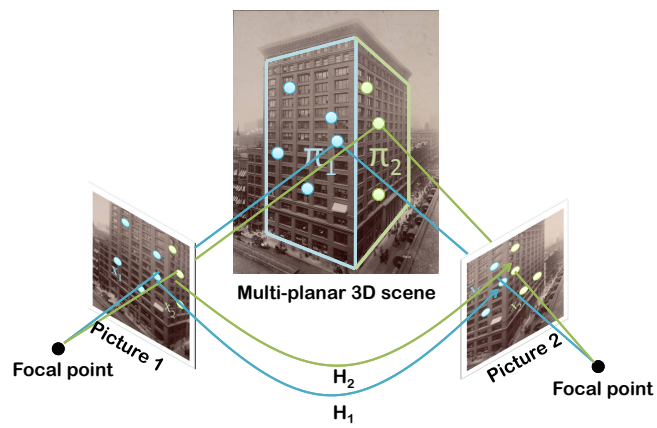

Figure 1: An example illustrating the emergence of homographies. Homographies encapsulate the relationship between the coordinates of the images, taken from two different views, of flat surfaces in the 3D scene. The two sides of the building can be viewed as being parts of two planes, $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$. The two planes give rise to two homographies represented by matrices, $\mathbf{H}_1$ and $\mathbf{H}_2$. The image coordinates of the side of the building associated with $\boldsymbol{\pi}_i$ in one view can be mapped with the aid of $\mathbf{H}_i$ to the image coordinates of the same side in the other view. In particular, given the coordinates of a window located on $\boldsymbol{\pi}_i$ in the first view, one can determine—by exploiting $\mathbf{H}_i$—the coordinates of the same window in the second view.

Interestingly, when properly exploited, the two different characterisations of multiple homography matrices lead to two different sets of compatibility constraints. The situation here resembles that arising in regard to the trifocal tensor from multiple view geometry: the entries of the tensor are amenable to several sets of constraints [18, 24] and each such set is derived from a specific characterisation of what constitutes a valid trifocal tensor among tensors of an algebraically relevant type.

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2. Two approaches

A set of 3×3 invertible matrices constitutes a set of *compatible* homography matrices if the matrices of the set represent homographies induced by multiple planes between two views. Mathematically, the compatibility requirement can be captured by two seemingly different definitions. We describe these next.

2.1. Latent variables

Consider two cameras with rank-3 camera matrices $\mathbf{P}_1 = [\mathbf{A}_1, \mathbf{b}_1]$ and $\mathbf{P}_2 = [\mathbf{A}_2, \mathbf{b}_2]$, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are 3×3 matrices, and $\mathbf{b}_1$ and $\mathbf{b}_2$ are length-3 vectors. Assume additionally that the first camera is finite, meaning that $\mathbf{A}_1$ is invertible. Select a plane in the 3D scene. The plane is determined by a length-4 vector $\boldsymbol{\pi} = [\mathbf{n}^\top, -d]^\top$, where $\mathbf{n}$ is the outward-pointing unit normal to the plane and d is the distance from the plane to the origin of the world coordinate system. If points in the scene are represented by length-4 homogeneous vectors $\mathbf{x}$, then the points that belong to the plane are exactly the solutions of the equation $\boldsymbol{\pi}^\top \mathbf{x} = 0$. The plane induces a planar homography between the first and second views. The homography is described by the 3×3 matrix

$$\mathbf{H} = w\mathbf{A} + \mathbf{b}\mathbf{v}^\top,$$

where

$$\mathbf{A} = \mathbf{A}_2\mathbf{A}_1^{-1}, \qquad \mathbf{b} = \mathbf{b}_2 - \mathbf{A}_2\mathbf{A}_1^{-1}\mathbf{b}_1, \tag{1}$$

$$w = d + \mathbf{n}^\top\mathbf{A}_1^{-1}\mathbf{b}_1, \qquad \mathbf{v} = \mathbf{A}_1^{-\top}\mathbf{n} \tag{2}$$

(cf. [11, 29]). Here, as a moment's reflection shows, $\mathbf{A}$ represents the *infinite homography* between the two views and $\mathbf{b}$ represents the *epipole* in the second image.

Slightly generalising the above setting, let us keep the camera matrices intact and suppose now that we are given a collection of I planes in the scene. For each $i = 1, \ldots, I$, let $\boldsymbol{\pi}_i = [\mathbf{n}_i^\top, -d_i]^\top$ describe the i-th plane from the collection. Then, for each $i = 1, \ldots, I$, the i-th homography between the two views induced by the i-th plane is specified by the 3×3 matrix

$$\mathbf{H}_i = w_i\mathbf{A} + \mathbf{b}\mathbf{v}_i^\top, \tag{3}$$

where $\mathbf{A}$ and $\mathbf{b}$ are given in (1), and w_i and $\mathbf{v}_i$ are given by

$$w_i = d_i + \mathbf{n}_i^\top\mathbf{A}_1^{-1}\mathbf{b}_1 \quad \text{and} \quad \mathbf{v}_i = \mathbf{A}_1^{-\top}\mathbf{n}_i. \tag{4}$$

With the representation (3) at hand, $\mathbf{A}$, $\mathbf{b}$, $\mathbf{v}_1, \ldots, \mathbf{v}_I$, $w_1, \ldots, w_I$ can be viewed as *latent variables* that are shared between the homography matrices.

If invertible matrices $\mathbf{H}_1, \ldots, \mathbf{H}_I$ satisfy (3) for some collection $\mathbf{A}$, $\mathbf{b}$, $\mathbf{v}_1, \ldots, \mathbf{v}_I$, $w_1, \ldots, w_I$, then one can always interpret the $\mathbf{H}_i$ as homography matrices induced by a set of planes in the scene. Indeed, if we let $\mathbf{P}_1 = [\mathbf{I}_3, \mathbf{0}]$, where $\mathbf{I}_3$ denotes the 3×3 identity matrix, and $\mathbf{P}_2 = [\mathbf{A}, \mathbf{b}]$, and if, for each $i = 1, \ldots, I$, we let the i-th plane in the 3D scene be described by $\boldsymbol{\pi}_i = [\mathbf{v}_i^\top, -w_i]^\top$, then the corresponding homography matrices described jointly by (1), (3), and (4) will coincide with the original matrices $\mathbf{H}_i$. In this argument, for $\mathbf{P}_2$ to be a legitimate camera matrix, the rank of $\mathbf{P}_2$ has to be three. This is always guaranteed when $\mathbf{A}$ is invertible. In turn, the requirement that $\mathbf{A}$ be invertible can always be met by a suitable choice of latent variables. More specifically, if $\mathbf{A}$ is singular, then the $\mathbf{H}_i$ can be expressed in terms of a different collection $\mathbf{A}'$, $\mathbf{b}'$, $\mathbf{v}'_1, \ldots, \mathbf{v}'_I$, $w'_1, \ldots, w'_I$ such that $\mathbf{A}'$ is invertible. Letting $\mathbf{P}_2 = [\mathbf{A}', \mathbf{b}']$ gives then a valid camera matrix. The passage from one set of latent variables to the other is described in more detail in Section 4.

Denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^n$ the set of length-n column vectors with real components. Let $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ matrices with real entries. The above observations inspire the following definition.

Definition 1. Let $\mathbf{H}_1, \ldots, \mathbf{H}_I \in \mathbb{R}^{3\times 3}$ be invertible matrices. Then the $\mathbf{H}_i$ form a set of compatible homography matrices if there exist a matrix $\mathbf{A} \in \mathbb{R}^{3\times 3}$, vectors $\mathbf{b}, \mathbf{v}_1, \ldots, \mathbf{v}_I \in \mathbb{R}^3$, and scalars $w_1, \ldots, w_I \in \mathbb{R}$ such that (3) holds for each $i = 1, \ldots, I$.

This definition was exploited, among others, in [4, 5, 7–9, 30, 31], and in [6] it was used to develop compatibility constraints for multiple homographies between two views.

2.2. Fundamental matrix

It is well known that any homography matrix $\mathbf{H}$ representing a homography induced by a plane between two views and the fundamental matrix $\mathbf{F}$ between the same two views are related by the system of equations

$$\mathbf{H}^\top\mathbf{F} + \mathbf{F}^\top\mathbf{H} = \mathbf{0} \tag{5}$$

(see [22, 23]). In the multiview geometry literature, this system is often referred to simply as the *compatibility constraint.*

Let $\mathbb{R}^{n\times m}_r$ denote the set of $n \times m$ real matrices with rank exactly r. Bearing in mind that a fundamental matrix is always of rank two, one can take (5) as the basis for the second definition of multi-homography compatibility as follows.

Definition 2. Let $\mathbf{H}_1, \ldots, \mathbf{H}_I \in \mathbb{R}^{3\times 3}$ be invertible matrices. Then the $\mathbf{H}_i$ form a set of compatible homography matrices if there exists a matrix $\mathbf{F} \in \mathbb{R}^{3\times 3}_2$ such that

$$\mathbf{H}_i^\top\mathbf{F} + \mathbf{F}^\top\mathbf{H}_i = \mathbf{0} \tag{6}$$

for each $i = 1, \ldots, I$.

When (6) holds, we say that the $\mathbf{H}_i$ are compatible among themselves and, also, that they are compatible with $\mathbf{F}$.

The definition above arises naturally in light of the discussion carried out in [14, Section 12.1]. Our forthcoming paper [33] will present compatibility constraints for multiple homographies based on this definition.

As a side remark, we point out that if an invertible $\mathbf{H} \in \mathbb{R}^{3\times 3}$ and a non-zero $\mathbf{F} \in \mathbb{R}^{3\times 3}$ are such that (5) holds, then $\mathbf{F}$ is necessarily of rank two. Indeed, condition (5) can be rephrased as the requirement that $\mathbf{H}^\top\mathbf{F}$ be antisymmetric. Since the rank of an antisymmetric matrix is always even [16, Corollary 4.4.19], the rank of $\mathbf{H}^\top\mathbf{F}$ is either zero or two. Since, just like $\mathbf{H}$, $\mathbf{H}^\top$ is invertible and $\mathbf{F}$ is non-zero, it follows that $\mathbf{H}^\top\mathbf{F}$ is non-zero and hence of rank two. Now, the rank of $\mathbf{F}$ is the same as that of $\mathbf{H}^\top\mathbf{F}$, owing to the invertibility of $\mathbf{H}^\top$. Therefore, finally, the rank of $\mathbf{F}$ is two. In light of this remark, it is clear that the matrix $\mathbf{F}$ in Definition 2 may just be assumed to be non-zero—it will automatically have to be of rank two. As revealed in [33], this has a fundamental bearing on the development of constraints for multiple homographies based on Definition 2.

3. Auxiliary results

Before showing that the two definitions of compatible homography matrices are equivalent, we present some auxiliary results that we state below in the form of two theorems.

For a length-3 vector $\mathbf{a} = [a_1, a_2, a_3]^\top$, let $[\mathbf{a}]_\times$ denote the 3×3 anti-symmetric matrix given by

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

We recall the following fundamental property of the matrix $[\mathbf{a}]_\times$: if we let $\times$ denote the cross products of length-3 vectors, then $\mathbf{a} \times \mathbf{y} = [\mathbf{a}]_\times \mathbf{y}$ for each $\mathbf{y} \in \mathbb{R}^3$.

Theorem 1. *If* $\mathbf{F}$ *is a* 3×3 *matrix of rank two, then there exist an invertible* 3×3 *matrix* $\mathbf{A}$ *and a non-zero length-*3 *vector* $\mathbf{b}$ *such that*

$$\mathbf{F} = [\mathbf{b}]_\times \mathbf{A}. \tag{7}$$

Conversely, if $\mathbf{F} \in \mathbb{R}^{3\times 3}$ *satisfies* (7) *for some invertible* $\mathbf{A} \in \mathbb{R}^{3\times 3}$ *and some* $\mathbf{b} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$*, then* $\mathbf{F}$ *has rank two.*

Proof. The proof is in two parts.

Necessity. Suppose that $\mathbf{F} \in \mathbb{R}_2^{3\times 3}$. Then the null space of $\mathbf{F}$ is one-dimensional and is spanned by a non-zero length-3 vector $\mathbf{a}$, and, likewise, the null space of $\mathbf{F}^\top$ is one-dimensional and is spanned by a non-zero length-3 vector $\mathbf{b}$. In particular, $\mathbf{F}^\top \mathbf{b} = \mathbf{0}$, which is equivalent to

$$\mathbf{b}^\top \mathbf{F} = \mathbf{0}^\top. \tag{8}$$

The last equation together with the identity

$$([\mathbf{b}]_\times)^2 = \mathbf{b}\mathbf{b}^\top - \|\mathbf{b}\|^2 \mathbf{I}_3,$$

where $\|\mathbf{b}\| = (\mathbf{b}^\top \mathbf{b})^{1/2}$ is the Euclidean norm of $\mathbf{b}$ (see e.g. [2, Fact 3.10.1, *iv)*]), implies that

$$([\mathbf{b}]_\times)^2 \mathbf{F} = \mathbf{b}\mathbf{b}^\top \mathbf{F} - \|\mathbf{b}\|^2 \mathbf{F} = -\|\mathbf{b}\|^2 \mathbf{F}.$$

With $\mathbf{A}_0 = -\|\mathbf{b}\|^{-2} [\mathbf{b}]_\times \mathbf{F}$, the equality of the leftmost and rightmost terms can be rewritten as

$$\mathbf{F} = [\mathbf{b}]_\times \mathbf{A}_0. \tag{9}$$

Now select $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v}^\top \mathbf{a} \neq 0$ and let

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{b}\mathbf{v}^\top.$$

Then, as $[\mathbf{b}]_\times \mathbf{b} = \mathbf{0}$, we have

$$[\mathbf{b}]_\times \mathbf{A} = [\mathbf{b}]_\times \mathbf{A}_0,$$

and this together with (9) implies (7).

It remains to show that $\mathbf{A}$ is invertible. Suppose, contrariwise, that $\mathbf{A}$ is not invertible. Then there exists an $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$, or equivalently,

$$\mathbf{A}_0 \mathbf{x} + (\mathbf{v}^\top \mathbf{x}) \mathbf{b} = \mathbf{0}. \tag{10}$$

Pre-multiplying this equation by $\mathbf{b}^\top$ and taking into account that $\mathbf{b}^\top \mathbf{A}_0 = \mathbf{0}^\top$ (since $\mathbf{b}^\top [\mathbf{b}]_\times = \mathbf{0}^\top$), we obtain $(\mathbf{v}^\top \mathbf{x})\|\mathbf{b}\|^2 = 0$ and further

$$\mathbf{v}^\top \mathbf{x} = 0. \tag{11}$$

Consequently, (10) reduces to $\mathbf{A}_0 \mathbf{x} = \mathbf{0}$, and this means that $[\mathbf{b}]_\times \mathbf{F}\mathbf{x} = \mathbf{0}$, or, what is the same, $\mathbf{b} \times \mathbf{F}\mathbf{x} = \mathbf{0}$. From this we infer that there exists a scalar α such that

$$\mathbf{F}\mathbf{x} = \alpha \mathbf{b}. \tag{12}$$

Now $\mathbf{b}^\top \mathbf{F}\mathbf{x} = \alpha \|\mathbf{b}\|^2$ and further, in view of (8), $\alpha \|\mathbf{b}\|^2 = 0$. Consequently, $\alpha = 0$, and so, by (12), $\mathbf{F}\mathbf{x} = \mathbf{0}$. Since the null space of $\mathbf{F}$ is spanned by $\mathbf{a}$, we have $\mathbf{x} = \beta \mathbf{a}$ for some non-zero $\beta \in \mathbb{R}$. It now follows from (11) that $\mathbf{v}^\top \mathbf{a} = 0$, contrary to the assumption about the choice of $\mathbf{v}$. This proves that $\mathbf{A}$ is indeed invertible.

Sufficiency. Suppose that $\mathbf{F} \in \mathbb{R}^{3\times 3}$ satisfies (7) for some invertible $\mathbf{A} \in \mathbb{R}^{3\times 3}$ and some $\mathbf{b} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Since $\mathbf{A}$ has full rank, $\mathbf{F}$ has the same rank as $[\mathbf{b}]_\times$, and since $[\mathbf{b}]_\times$ has rank two, so too does $\mathbf{F}$.

Theorem 2. *Let* $\mathbf{R} \in \mathbb{R}^{3\times 3}$ *and* $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ *be such that*

$$[\mathbf{x}]_\times \mathbf{R} = \mathbf{R}^\top [\mathbf{x}]_\times. \tag{13}$$

Then there exist $w \in \mathbb{R}$ *and* $\mathbf{y} \in \mathbb{R}^3$ *such that* $\mathbf{R} = w\mathbf{I}_3 + \mathbf{x}\mathbf{y}^\top$.

Proof. We split the proof into a number of steps.

Step 1. Denote by $\mathbb{N}_0$ the set of non-negative integers and by $\mathbb{N}$ the set of positive integers. We first establish that

$$[\mathbf{x}]_\times \mathbf{R}^n = (\mathbf{R}^\top)^n [\mathbf{x}]_\times \tag{14}$$

for each $n \in \mathbb{N}_0$. Indeed, if $n = 0$, then (14) holds vacuously, as then $\mathbf{R}^n = (\mathbf{R}^\top)^n = \mathbf{I}_3$; and if (14) holds for some $n \in \mathbb{N}_0$, then, by (13),

$$\begin{aligned} [\mathbf{x}]_\times \mathbf{R}^{n+1} &= [\mathbf{x}]_\times \mathbf{R}^n \cdot \mathbf{R} = (\mathbf{R}^\top)^n [x]_\times \mathbf{R} \\ &= (\mathbf{R}^\top)^n \cdot \mathbf{R}^\top [\mathbf{x}]_\times = (\mathbf{R}^\top)^{n+1} [\mathbf{x}]_\times, \end{aligned}$$

showing that (14) also holds for $n + 1$. The argument is now completed by induction.

Step 2. Since the theorem trivially holds if $\mathbf{R}$ is a null matrix (with $w = 0$ and $\mathbf{y} = \mathbf{0}$), we may safely assume that $\mathbf{R}$ is non-null. With this assumption in force, we next consider the auxiliary matrix $\mathbf{I}_3 + (2\|\mathbf{R}\|)^{-1}\mathbf{R}$, where $\|\cdot\|$ is any submultiplicative matrix norm, say the Frobenius norm, in which case $\|\mathbf{S}\| = \left(\sum_{i,j=1}^3 |s_{ij}|^2\right)^{1/2}$ whenever $\mathbf{S} = [s_{ij}]_{1\le i,j\le 3} \in \mathbb{R}^{3\times 3}$. We shall make use of one particular square root of this matrix which we define next.[1]

For any $\alpha \in \mathbb{R}$, let $\binom{\alpha}{0} = 1$ and

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

[1] Not every matrix has a matrix square root, for example $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is a matrix without a square root.

for each $n \in \mathbb{N}$. It is standard that, for all $\alpha \in \mathbb{R}$, the infinite series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

converges absolutely for x in the open interval $(-1, 1)$, and, moreover,

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = (1+x)^{\alpha} \tag{15}$$

for every $x \in (-1, 1)$ (see e.g. [15, Thm. 7.25]). Let

$$\mathbf{T} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (2\|\mathbf{R}\|)^{-n} \mathbf{R}^n. \tag{16}$$

Since the norm of $(2\|\mathbf{R}\|)^{-1}\mathbf{R}$ equals $1/2$, it follows by the sub-multiplicativity of the adopted matrix norm that the norm of $(2\|\mathbf{R}\|)^{-n}\mathbf{R}^n$ is no greater than 2^{-n} for each $n \in \mathbb{N}_0$, and so

$$\sum_{n=0}^{\infty} \left| \binom{\frac{1}{2}}{n} \right| \left\| (2\|\mathbf{R}\|)^{-n} \mathbf{R}^n \right\| \leq \sum_{n=0}^{\infty} \left| \binom{\frac{1}{2}}{n} \right| 2^{-n} < \infty.$$

We see that the matrix-valued series on the right-hand side of (16) converges absolutely. Formula (15) suggests that $\mathbf{T}$ is a square root of $\mathbf{I}_3 + (2\|\mathbf{R}\|)^{-1}\mathbf{R}$,

$$\mathbf{T}^2 = \mathbf{I}_3 + (2\|\mathbf{R}\|)^{-1}\mathbf{R}. \tag{17}$$

To prove that this indeed is the case, note that

$$\begin{aligned} 1 + x &= (1+x)^{1/2}(1+x)^{1/2} \\ &= \left[\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \right] \left[\sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} x^m \right] \\ &= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{k} \binom{\frac{1}{2}}{n} \binom{\frac{1}{2}}{k-n} \right] x^k \end{aligned}$$

for every $x \in (-1, 1)$. Comparing the coefficients at the powers of x, we see that

$$\sum_{n=0}^{k} \binom{\frac{1}{2}}{n} \binom{\frac{1}{2}}{k-n} = \begin{cases} 1 & \text{if } k = 0, 1 \\ 0 & \text{if } k = 2, 3, \ldots \end{cases}.$$

With this formula, we now have

$$\begin{aligned} \mathbf{T}^2 &= \left[\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (2\|\mathbf{R}\|)^{-n} \mathbf{R}^n \right] \left[\sum_{m=0}^{\infty} \binom{\frac{1}{2}}{m} (2\|\mathbf{R}\|)^{-m} \mathbf{R}^m \right] \\ &= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{k} \binom{\frac{1}{2}}{n} \binom{\frac{1}{2}}{k-n} \right] (2\|\mathbf{R}\|)^{-k} \mathbf{R}^k \\ &= \mathbf{I}_3 + (2\|\mathbf{R}\|)^{-1}\mathbf{R}, \end{aligned}$$

so (17) is established.

We next show that $\mathbf{T}$ is invertible with the inverse given by

$$\mathbf{T}^{-1} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (2\|\mathbf{R}\|)^{-n} \mathbf{R}^n. \tag{18}$$

To this end, note that

$$\begin{aligned} 1 &= (1+x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} \\ &= \left[\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n \right] \left[\sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} x^m \right] \\ &= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{k} \binom{\frac{1}{2}}{n} \binom{-\frac{1}{2}}{k-n} \right] x^k \end{aligned}$$

for every $x \in (-1, 1)$. Consequently,

$$\sum_{n=0}^{k} \binom{\frac{1}{2}}{n} \binom{-\frac{1}{2}}{k-n} = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \in \mathbb{N}. \end{cases}$$

This formula implies that if we denote the matrix defined by the right-hand side of (18) by $\mathbf{U}$, then

$$\begin{aligned} \mathbf{TU} &= \left[\sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (2\|\mathbf{R}\|)^{-n} \mathbf{R}^n \right] \left[\sum_{m=0}^{\infty} \binom{-\frac{1}{2}}{m} (2\|\mathbf{R}\|)^{-m} \mathbf{R}^m \right] \\ &= \sum_{k=0}^{\infty} \left[\sum_{n=0}^{k} \binom{\frac{1}{2}}{n} \binom{-\frac{1}{2}}{k-n} \right] (2\|\mathbf{R}\|)^{-k} \mathbf{R}^k \\ &= \mathbf{I}_3. \end{aligned}$$

Thus $\mathbf{T}$ and $\mathbf{U}$ are both invertible and they are inverses of each other, which in particular yields (18).

We finally note that it immediately follows from (14) and $\mathbf{T}$ being a power series in $\mathbf{R}$ that

$$[\mathbf{x}]_\times \mathbf{T} = \mathbf{T}^\top [\mathbf{x}]_\times. \tag{19}$$

Step 3. We now show that $\mathbf{x}$ is an eigenvector of $\mathbf{T}$. Indeed, since, by (19), $([\mathbf{x}]_\times)^2\mathbf{T} = [\mathbf{x}]_\times \mathbf{T}^\top [\mathbf{x}]_\times$ and since $[\mathbf{x}]_\times \mathbf{x} = \mathbf{0}$, we have

$$([\mathbf{x}]_\times)^2 \mathbf{T}\mathbf{x} = \mathbf{0}.$$

This together with

$$([\mathbf{x}]_\times)^2 = \mathbf{x}\mathbf{x}^\top - \|\mathbf{x}\|^2 \mathbf{I}_3 \tag{20}$$

implies that

$$(\mathbf{x}^\top \mathbf{T}\mathbf{x})\mathbf{x} - \|\mathbf{x}\|^2 \mathbf{T}\mathbf{x} = \mathbf{0}.$$

Consequently, letting $\lambda = \|\mathbf{x}\|^{-2}\mathbf{x}^\top \mathbf{T}\mathbf{x}$, we have $\mathbf{T}\mathbf{x} = \lambda \mathbf{x}$.

Critically, since $\mathbf{T}$ is invertible, it follows that $\lambda \neq 0$ and we also have

$$\mathbf{T}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}. \tag{21}$$

Step 4. Recall that

$$\mathbf{T}^\top [\mathbf{T}\mathbf{z}]_\times \mathbf{T} = (\det \mathbf{T})[\mathbf{z}]_\times$$

for any $\mathbf{z} \in \mathbb{R}^3$ (see e.g. [2, Fact 3.10.1, *xxxvi)*]). Using this with $\mathbf{z} = \mathbf{T}^{-1}\mathbf{x}$ and taking into account (13) and (21), we see that

$$\begin{aligned} [\mathbf{x}]_\times \mathbf{T}^2 &= \mathbf{T}^\top [\mathbf{x}]_\times \mathbf{T} = \mathbf{T}^\top [\mathbf{T}\mathbf{T}^{-1}\mathbf{x}]_\times \mathbf{T} \\ &= (\det \mathbf{T})[\mathbf{T}^{-1}\mathbf{x}]_\times = \lambda^{-1}(\det \mathbf{T})[\mathbf{x}]_\times. \end{aligned}$$

Hence

$$([\mathbf{x}]_\times)^2 \mathbf{T}^2 = \lambda^{-1}(\det \mathbf{T})([\mathbf{x}]_\times)^2.$$

Invoking (20) again, we can rewrite the last equation as

$$\mathbf{x}\mathbf{x}^\top\mathbf{T}^2 - \|\mathbf{x}\|^2\mathbf{T}^2 = \lambda^{-1}(\det\mathbf{T})(\mathbf{x}\mathbf{x}^\top - \|\mathbf{x}\|^2\mathbf{I}_3). \tag{22}$$

With

$$\mathbf{u} = \|\mathbf{x}\|^{-2}((\mathbf{T}^2)^\top\mathbf{x} - \lambda^{-1}(\det\mathbf{T})\mathbf{x})$$

and

$$v = \lambda^{-1}(\det\mathbf{T}),$$

(22) now becomes

$$\mathbf{T}^2 = v\mathbf{I}_3 + \mathbf{x}\mathbf{u}^\top.$$

Taking into account (17), we see that

$$\mathbf{I}_3 + (2\|\mathbf{R}\|)^{-1}\mathbf{R} = v\mathbf{I}_3 + \mathbf{x}\mathbf{u}^\top. \tag{23}$$

Finally, if we let

$$w = 2\|\mathbf{R}\|(v-1) \quad \text{and} \quad \mathbf{y} = 2\|\mathbf{R}\|\mathbf{u},$$

then (23) can be rewritten as $\mathbf{R} = w\mathbf{I}_3 + \mathbf{x}\mathbf{y}^\top$, which is the required representation.

4. Equivalence of the definitions

The argument for establishing the equivalence of Definitions 1 and 2 splits into two parts.

Definition 1 implies Definition 2. Let $\mathbf{H}_1, \ldots, \mathbf{H}_I \in \mathbb{R}^{3\times3}$ be invertible matrices satisfying (3) for $\mathbf{A} \in \mathbb{R}^{3\times3}$, $\mathbf{b}, \mathbf{v}_1, \ldots, \mathbf{v}_I \in \mathbb{R}^3$, and $w_1, \ldots, w_I \in \mathbb{R}$. We start with the claim that $\mathbf{A}$ may safely be assumed invertible—in fact $\mathbf{A}$ may be assumed equal to any of the invertible matrices $\mathbf{H}_i$.

First note that $w_i \neq 0$ for each $i = 1, \ldots, I$. Indeed, if $w_i = 0$ held for some i, then $\mathbf{H}_i$ would be equal to $\mathbf{b}\mathbf{v}_i^\top$ and hence would be of rank one, contravening the assumption that $\mathbf{H}_i$ (and any of the other homography matrices for that matter) is invertible. Next observe that if α and β are non-zero scalars and $\mathbf{c}$ is a length-3 vector, then, alongside the representation (3), we also have the representation

$$\mathbf{H}_i = w_i'\mathbf{A}' + \mathbf{b}'\mathbf{v}_i'^\top$$

for each $i = 1, \ldots, I$, where

$$\mathbf{A}' = \beta\mathbf{A} + \mathbf{b}\mathbf{c}^\top, \qquad \mathbf{b}' = \alpha\mathbf{b},$$
$$\mathbf{v}_i' = \alpha^{-1}\mathbf{v}_i - \alpha^{-1}\beta^{-1}w_i\mathbf{c}, \qquad w_i' = \beta^{-1}w_i.$$

We now exploit this last observation first by fixing $i_0 \in \{1, \ldots, I\}$ arbitrarily and next by letting $\alpha = 1$, $\beta = w_{i_0}$, and $\mathbf{c} = \mathbf{v}_{i_0}$. Critically, by our first observation, β is necessarily non-zero. Moreover, we have $\mathbf{A}' = \mathbf{H}_{i_0}$ and further $\mathbf{H}_i = w_i'\mathbf{H}_{i_0} + \mathbf{b}\mathbf{v}_i'^\top$ with $w_i' = w_{i_0}^{-1}w_i$ and $\mathbf{v}_i' = \mathbf{v}_i - w_{i_0}^{-1}w_i\mathbf{v}_{i_0}$ for each $i = 1, \ldots, I$ (for $i = i_0$, the equality is vacuously true). This establishes the claim.

We assume henceforth that $\mathbf{A}$ is invertible. Suppose first that $\mathbf{b} \neq \mathbf{0}$ and let

$$\mathbf{F} = [\mathbf{b}]_\times\mathbf{A}.$$

Then, as $[\mathbf{b}]_\times$ has rank two and $\mathbf{A}$ has full rank, $\mathbf{F}$ has rank two. Fix $i \in \{1, \ldots, I\}$ arbitrarily. Then, since $\mathbf{b}^\top[\mathbf{b}]_\times = \mathbf{0}^\top$, we have

$$\mathbf{H}_i^\top\mathbf{F} = (w_i\mathbf{A}^\top + \mathbf{v}_i\mathbf{b}^\top)[\mathbf{b}]_\times\mathbf{A} = w_i\mathbf{A}^\top[\mathbf{b}]_\times\mathbf{A}$$

and, since $([\mathbf{b}]_\times)^\top = -[\mathbf{b}]_\times$ and $[\mathbf{b}]_\times\mathbf{b} = \mathbf{0}$, we also have

$$\mathbf{F}^\top\mathbf{H}_i = -\mathbf{A}^\top[\mathbf{b}]_\times(w_i\mathbf{A} + \mathbf{b}\mathbf{v}_i^\top) = -w_i\mathbf{A}^\top[\mathbf{b}]_\times\mathbf{A}.$$

This immediately implies (6), showing that the $\mathbf{H}_i$ are compatible according to Definition 2.

Suppose now that $\mathbf{b} = \mathbf{0}$. Then $\mathbf{H}_i = w_i\mathbf{A}$ for each $i = 1, \ldots, I$. Pick $\mathbf{b}' \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ arbitrarily and let $\mathbf{F} = [\mathbf{b}']_\times\mathbf{A}$. Then, clearly, $\mathbf{F}$ has rank two. Moreover, for each $i = 1, \ldots, I$,

$$\mathbf{H}_i^\top\mathbf{F} = w_i\mathbf{A}^\top[\mathbf{b}']_\times\mathbf{A}$$

and

$$\mathbf{F}^\top\mathbf{H}_i = -w_i\mathbf{A}^\top[\mathbf{b}']_\times\mathbf{A},$$

and this, as before, implies (6). We conclude that the $\mathbf{H}_i$ are compatible in the sense of Definition 2 also in the current case.

Definition 2 implies Definition 1. Let $\mathbf{H}_1, \ldots, \mathbf{H}_I \in \mathbb{R}^{3\times3}$ be invertible matrices and let $\mathbf{F} \in \mathbb{R}_2^{3\times3}$ be such that (6) holds for each $i = 1, \ldots, I$. Then, firstly, Theorem 1 ensures that (7) holds for some invertible $\mathbf{A} \in \mathbb{R}^{3\times3}$ and some $\mathbf{b} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$. Secondly, as $([\mathbf{b}]_\times)^\top = -[\mathbf{b}]_\times$, it follows from (6) that

$$\mathbf{H}_i^\top[\mathbf{b}]_\times\mathbf{A} = \mathbf{A}^\top[\mathbf{b}]_\times\mathbf{H}_i \tag{24}$$

for each $i = 1, \ldots, I$. Now, fix $i \in \{1, \ldots, I\}$ arbitrarily and let

$$\mathbf{R}_i = \mathbf{H}_i\mathbf{A}^{-1}.$$

Pre-multiplying both sides of (24) by $\mathbf{A}^{-\top}$ and post-multiplying both sides of the same equation by $\mathbf{A}^{-1}$, we find that

$$\mathbf{R}_i^\top[\mathbf{b}]_\times = [\mathbf{b}]_\times\mathbf{R}_i.$$

By Theorem 2, there exist $w_i \in \mathbb{R}$ and $\mathbf{y}_i \in \mathbb{R}^3$ such that

$$\mathbf{R}_i = w_i\mathbf{I}_3 + \mathbf{b}\mathbf{y}_i^\top.$$

Post-multiplying both sides of this equation by $\mathbf{A}$ yields

$$\mathbf{H}_i = w_i\mathbf{A} + \mathbf{b}\mathbf{y}_i^\top\mathbf{A},$$

or equivalently,

$$\mathbf{H}_i = w_i\mathbf{A} + \mathbf{b}\mathbf{v}_i^\top,$$

where $\mathbf{v}_i = \mathbf{A}^\top\mathbf{y}_i$. We conclude that the $\mathbf{H}_i$ are compatible in the sense of Definition 1.

5. Verification

We verify the validity of our equivalence result on a concrete example depicted in Figure 2. The scene is represented in a right-handed world coordinate system and consists of points on two planar surfaces $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$, where

$$\boldsymbol{\pi}_1 = [\mathbf{n}_1^\top, -d_1]^\top, \qquad \boldsymbol{\pi}_2 = [\mathbf{n}_2^\top, -d_2]^\top,$$
$$= [\,1\ 0\ 0\ 0\,]^\top, \qquad = [\,0\ 1\ 0\ {-2000}\,]^\top.$$

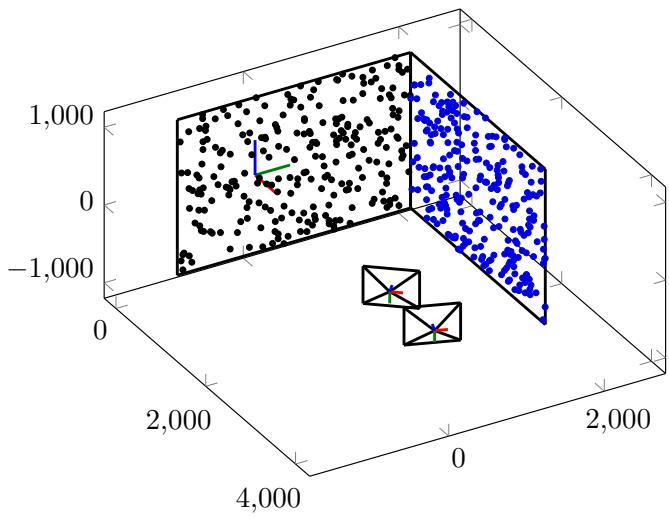

Figure 2: A synthetic 3D scene illustrating two planar surfaces and two cameras. Blue and black markers depict points that lie on the first and second plane, respectively. Large red, green and blue line segments represent the global Cartesian coordinate system of the scene, whereas short red, green and blue line segments represent the local Cartesian coordinate frames of the two cameras.

The two camera projection matrices are

$$\begin{aligned}\mathbf{P}_1 &= \mathbf{K}_1\mathbf{R}_1[\mathbf{I}_3, -\mathbf{t}_1]\\ &= \begin{bmatrix} 100 & 0 & 0\\ 0 & 100 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 0.642788 & 0.766044 & 0\\ 0 & 0 & -1\\ -0.766044 & 0.642788 & 0\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -3000\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}_2 &= \mathbf{K}_2\mathbf{R}_2[\mathbf{I}_3, -\mathbf{t}_2]\\ &= \begin{bmatrix} 150 & 0 & 0\\ 0 & 150 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 0.34202 & 0.939693 & 0\\ 0 & 0 & -1\\ -0.939693 & 0.34202 & 0\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -4000\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},\end{aligned}$$

where, for $k = 1, 2$, the length-3 translation vector $\mathbf{t}_k$ and the 3×3 rotation matrix $\mathbf{R}_k$ represent the Euclidean transformation between the k-th camera and the world coordinate system, and the 3×3 upper triangular calibration matrix $\mathbf{K}_k$ encodes the internal parameters of the k-th camera. Applying equations (3) and (4) yields two concomitant homography matrices, namely

$$\begin{aligned}\mathbf{H}_1 &= w_1\mathbf{A} + \mathbf{b}\mathbf{v}_1^\top\\ &= (d_1 - \mathbf{n}_1^\top\mathbf{t}_1)\mathbf{K}_2\mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{K}_1^{-1} + \mathbf{K}_2\mathbf{R}_2(\mathbf{t}_1 - \mathbf{t}_2)(\mathbf{K}_1^{-\top}\mathbf{R}_1\mathbf{n}_1)^\top\\ &= \begin{bmatrix} -4558.39 & 0 & -1.14609e5\\ 0 & -4500.0 & 0\\ 16.3008 & 0 & -3538.92\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}_2 &= w_2\mathbf{A} + \mathbf{b}\mathbf{v}_2^\top\\ &= (d_2 - \mathbf{n}_2^\top\mathbf{t}_1)\mathbf{K}_2\mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{K}_1^{-1} + \mathbf{K}_2\mathbf{R}_2(\mathbf{t}_1 - \mathbf{t}_2)(\mathbf{K}_1^{-\top}\mathbf{R}_1\mathbf{n}_2)^\top\\ &= \begin{bmatrix} 2426.07 & 0 & 69629.1\\ 0 & 3000.0 & 0\\ 0.35806 & 0 & 2483.41\end{bmatrix}.\end{aligned}$$

Finally, the fundamental matrix is given by the relation

$$\begin{aligned}\mathbf{F} &= [\mathbf{K}_2\mathbf{R}_2(\mathbf{t}_1 - \mathbf{t}_2)]_\times\mathbf{K}_2\mathbf{R}_2\mathbf{R}_1^{-1}\mathbf{K}_1^{-1}\\ &= \begin{bmatrix} 0 & -1409.54 & 0\\ 1149.07 & 0 & 96418.1\\ 0 & -76954.5 & 0\end{bmatrix}.\end{aligned}$$

Direct substitution confirms that the fundamental and homography matrices satisfy equations in system (6).

6. Conclusion

We have presented two definitions of what constitutes a set of compatible homography matrices for the description of a set of homographies between two fixed views, and we have shown the equivalence of these definitions. As mentioned earlier, one of the two definitions was used, in [6], to develop a full set of compatibility constraints for multiple homography matrices. The result of the present paper opens an avenue for identification of another full set of compatibility constraints for multiple homography matrices, this time exploiting the other definition. It was already demonstrated experimentally that the set of compatibility constraints based on the first definition in question leads to more accurate estimates of multiple homography matrices from image measurements. Leaping forward, one can presume that the prospective new full sets of compatibility constraints will also result in enhancing the accuracy of estimation of multiple homographies from image measurements. While the levels of improvement of estimation accuracy might be on par in both cases, it is conceivable that the alternative set of constraints will lead to a numerically more effective optimisation method.

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