

# Eigenvalues of Almost Periodic Schrödinger Operators in $L^2(b\mathbb{R})$ are at most Double

WOJCIECH CHOJNACKI

*Institute of Applied Mathematics and Mechanics, University of Warsaw, ul. Banacha 2,  
00-913 Warsaw 59, Poland and School of Information Science and Technology,  
The Flinders University of South Australia, GPO Box 2100, Adelaide, SA 5001, Australia*

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**Abstract.** We prove that every eigenvalue of a Schrödinger operator with an almost periodic potential acting in the space of all square Haar-integrable functions on the Bohr compactification of  $\mathbb{R}$  has multiplicity not greater than 2.

**Key words.** Almost periodic Schrödinger operator, Bohr compactification, eigenvalue, multiplicity

**AMS subject classifications (1980).** 35J10, 35P99.

## 1. Introduction

It is well known that a formal Schrödinger operator with an almost periodic potential gives rise to two self-adjoint operators, one of which acts in the usual Hilbert space  $L^2(\mathbb{R}^n)$  and the other acts in the nonseparable Hilbert space  $L^2(b\mathbb{R}^n)$  of all square Haar-integrable functions on the Bohr compactification  $b\mathbb{R}^n$  of  $\mathbb{R}^n$ . These two operators are closely interrelated. Each of them can be represented as a restriction of an ultrapower of the other, which implies that the spectra of both operators coincide (cf. [3, 5]). Moreover, the so-called spectral duality theorem, whose presumably incomplete version appears in [1] and whose rigorous version appears in [4], implies the following. If the spectral measure of a smooth almost periodic Schrödinger operator in  $L^2(b\mathbb{R}^n)$  over a Borel subset of  $\mathbb{R}$  has pure point spectrum with finite multiplicity, then almost all (with respect to the Haar measure on  $b\mathbb{R}^n$ ) the spectral measures of Schrödinger operators in  $L^2(\mathbb{R}^n)$  whose potentials are the translates by elements of  $b\mathbb{R}^n$  of the initial almost periodic potential have, over the same set, purely continuous spectrum.

In this Letter, we show that every eigenvalue of an almost periodic (not necessarily smooth) Schrödinger operator in  $L^2(b\mathbb{R})$  has multiplicity not greater than 2. As a result, the above-mentioned consequence of the spectral duality theorem in the case where  $n = 1$  may be strengthened by the omission of the assumption about finite multiplicity.

## 2. The Result

Let  $\alpha$  be the canonical homomorphism from  $\mathbb{R}$  into  $b\mathbb{R}$ . For each  $\mu \in \mathbb{R}$ , let  $\chi_\mu$  denote the continuous character of  $b\mathbb{R}$  such that  $\chi_\mu(\alpha(x)) = e^{i\mu x}$  for all  $x \in \mathbb{R}$ . Let  $\lambda_{b\mathbb{R}}$  be the probabilistic Haar measure on  $b\mathbb{R}$ , and  $L^p(b\mathbb{R})$  ( $1 \leq p < +\infty$ ) be the Lebesgue spaces based on  $\lambda_{b\mathbb{R}}$ . Given  $F \in L^1(b\mathbb{R})$  and  $\mu \in \mathbb{R}$ , let  $\hat{F}(\mu)$  denote the  $\mu$ th Fourier coefficient of  $F$ , i.e.,

$$\hat{F}(\mu) = \int_{b\mathbb{R}} F \chi_{-\mu} \, d\lambda_{b\mathbb{R}}.$$

As is well known, the Fourier transformation  $F \rightarrow (\hat{F}(\mu))_{\mu \in \mathbb{R}}$  sets up a unitary isomorphism between  $L^2(b\mathbb{R})$  and  $l^2(\mathbb{R})$ .

Given a function  $f$  on  $b\mathbb{R}$  and  $t \in \mathbb{R}$ , let  $T_t f$  denote the translate of  $f$  by  $\alpha(t)$ , i.e.,

$$T_t f(\omega) = f(\omega + \alpha(t)) \quad (\omega \in b\mathbb{R}).$$

In each space  $L^p(b\mathbb{R})$  ( $1 \leq p < +\infty$ ), the operators  $\{T_t : t \in \mathbb{R}\}$  form a strongly continuous one-parameter group with infinitesimal generator  $D_p$ . If  $F$  is in the domain  $\mathcal{D}(D_p)$  of  $D_p$ , then  $D_p F(\mu) = i\mu \hat{F}(\mu)$  for each  $\mu \in \mathbb{R}$ . Hence, if  $D_p F = 0$ , then  $F$  is  $\lambda_{b\mathbb{R}}$ -essentially constant.  $\mathcal{D}(D_2)$  coincides with the subspace of those  $F$ 's in  $L^2(b\mathbb{R})$  for which  $\sum_{\mu \in \mathbb{R}} \mu^2 |\hat{F}(\mu)|^2 < +\infty$ . For  $p \neq 2$ , there is no similar characterization of  $\mathcal{D}(D_p)$ . If  $F \in \mathcal{D}(D_p)$  and  $G \in \mathcal{D}(D_q)$  with  $p^{-1} + q^{-1} = 1$ , then  $FG \in \mathcal{D}(D_1)$  and

$$D_1(FG) = (D_p F)G + FD_q G.$$

Let  $q$  be a real uniformly almost periodic function on  $\mathbb{R}$ . With  $q$  there is associated the Schrödinger operator

$$l = -\frac{d^2}{dx^2} + q$$

defined on the Sobolev space  $H^2(\mathbb{R})$ . Let  $Q$  be the unique continuous function on  $b\mathbb{R}$  such that  $Q(\alpha(x)) = q(x)$  for all  $x \in \mathbb{R}$ .  $Q$  gives rise to the analogue of  $l$  in  $L^2(b\mathbb{R})$ , namely the self-adjoint operator  $\mathcal{L} = -D_2^2 + Q$  defined on

$$\mathcal{D}(\mathcal{L}) = \left\{ F \in L^2(b\mathbb{R}) : \sum_{\mu \in \mathbb{R}} \mu^4 |\hat{F}(\mu)|^2 < +\infty \right\}.$$

We have the following

**THEOREM.** *Every eigenvalue of  $\mathcal{L}$  has multiplicity not greater than 2.*

*Proof.* Let  $X$  and  $Y$  be eigenvectors of  $\mathcal{L}$  with eigenvalue  $\lambda$ . Then

$$\begin{aligned} D_1((D_2 X)Y - XD_2 Y) &= (D_2^2 X)Y - X(D_2^2 Y) \\ &= (Q - \lambda)XY - X(Q - \lambda)Y = 0. \end{aligned}$$

Hence  $(D_2 X)Y - XD_2 Y$  is  $\lambda_{b\mathbb{R}}$ -essentially constant.

Suppose that  $X$  and  $Y$  are linearly independent. We claim that  $(D_2X)Y - XD_2Y$  is  $\lambda_{b\mathbb{R}}$ -essentially nonzero.

For each  $t \in \mathbb{R}$ , set

$$f(t) = (T_t X)Y - XT_t Y.$$

Since the functions

$$\mathbb{R} \ni t \rightarrow T_t X \in L^2(b\mathbb{R}) \quad \text{and} \quad \mathbb{R} \ni t \rightarrow T_t Y \in L^2(b\mathbb{R})$$

are twice norm-differentiable and

$$\frac{d^2 T_t X}{dt^2} = T_t D_2^2 X = (T_t Q - \lambda)T_t X$$

and

$$\frac{d^2 T_t Y}{dt^2} = (T_t Q - \lambda)T_t Y,$$

it follows that the function  $\mathbb{R} \ni t \rightarrow f(t) \in L^1(b\mathbb{R})$  is twice norm-differentiable and

$$\frac{d^2 f(t)}{dt^2} = (T_t Q - \lambda)f(t).$$

Of course,  $f(0) = 0$ . Should

$$\frac{df}{dt}(0) = (D_2 X)Y - XD_2 Y = 0,$$

then, by the standard uniqueness theorem for ordinary equations in Banach spaces (cf. [2, Remark 10.5.4]), we have that  $f(t) = 0$  for all  $t \in \mathbb{R}$ . Hence, for each  $\mu \in \mathbb{R}$  and each  $T > 0$ ,

$$X \frac{1}{2T} \int_{-T}^T (T_t Y) \chi_{-\mu} e^{-i\mu t} dt = Y \frac{1}{2T} \int_{-T}^T (T_t X) \chi_{-\mu} e^{-i\mu t} dt.$$

The flow  $\omega \rightarrow \omega + \alpha(t)$  ( $t \in \mathbb{R}$ ) on  $b\mathbb{R}$  is ergodic, so the mean ergodic theorem ensures that

$$\lim_{T \rightarrow \infty} \left\| \frac{1}{2T} \int_{-T}^T (T_t X) \chi_{-\mu} e^{-i\mu t} dt - \hat{X}(\mu) \right\|_2 = 0$$

and

$$\lim_{T \rightarrow \infty} \left\| \frac{1}{2T} \int_{-T}^T (T_t Y) \chi_{-\mu} e^{-i\mu t} dt - \hat{Y}(\mu) \right\|_2 = 0.$$

Thus  $Y\hat{X}(\mu) = X\hat{Y}(\mu)$ . Choosing  $\mu \in \mathbb{R}$  so that  $|\hat{X}(\mu)|^2 + |\hat{Y}(\mu)|^2 > 0$ , we see that  $X$  and  $Y$  are linearly dependent. This contradiction establishes the claim.

Now it is easily verified that if  $Z$  is an eigenvector of  $\mathcal{L}$  with eigenvalue  $\lambda$ , then  $Z = \alpha X + \beta Y$ , where  $\alpha$  and  $\beta$  are the  $\lambda_{b\mathbb{R}}$ -essential values of

$$[(D_2 Z)Y - ZD_2 Y][(D_2 X)Y - XD_2 Y]^{-1}$$

and

$$[(D_2 X)Z - XD_2 Z][(D_2 X)Y - XD_2 Y]^{-1},$$

respectively.

The proof is complete.

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