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Isolated points of spaces of homomorphisms from ordered AL-algebras

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Abstract

We study isolated points of spaces comprised of homomorphisms between a given ordered AL-algebra and a given unital normed algebra. An ordered AL-algebra is a complex AL-space and simultaneously a Banach algebra such that the positive cone associated with the underlying partial ordering is closed under multiplication, and such that the algebra norm restricted to the positive cone is multiplicative. The class of ordered AL-algebras contains—as particular subclasses—semigroup algebras, group algebras, and convolution algebras of integrable, even functions on groups. We determine isolated points for various spaces of homomorphisms from ordered AL-algebras, including specifically spaces of homomorphisms from algebras belonging to the three subclasses just mentioned. We also discuss certain properties of homomorphisms beyond isolability, which one is naturally led to consider in connection with isolated points of spaces of homomorphisms. By way of application, we exhibit several AL-algebras that are not pairwise isometrically algebra and order isomorphic.

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1. Introduction

Families of operators on a Banach space satisfying a functional equation have long been the object of intense studies. Semigroups, groups, and cosine families of operators are arguably the most prominent examples among these. Recently, the problem of identifying and studying isolated points of spaces formed by operator-valued solutions of a functional equation, acting on a common linear space, has attracted considerable interest. Various investigations have been made in regard to isolated points of spaces of strongly continuous semigroups and cosine families of operators [12], [13], [14], [17], [18], [29], [30], [76], integrated semigroups and operator sine functions [11], and operator-valued solutions of fractional evolution equations [33]. Inspired by these contributions, we embark in this memoir on a study of isolated points of spaces of homomorphisms from Banach algebras to normed algebras, with all homomorphisms in each relevant space sharing a common domain and a common codomain.

At the heart of our investigations is the observation that strongly continuous semigroups, groups, and cosine families of operators lead naturally to homomorphisms from certain Banach convolution algebras. To wit, if $\{\mathscr{S}(t)\}_{t\geq 0}$ is a uniformly bounded, strongly continuous one-parameter semigroup on a Banach space X, then

$$H(f) = \int_0^\infty f(s)\mathscr{S}(s)\,\mathrm{d}s \qquad (f \in L^1(\mathbb{R}^+))$$

defines a continuous homomorphism from $L^1(\mathbb{R}^+)$ into $\mathscr{L}(X)$. Here $L^1(\mathbb{R}^+)$ is the Banach algebra of equivalence classes of complex-valued, Lebesgue integrable functions on the non-negative half-line $\mathbb{R}^+$, with the convolution product

$$(f \star g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s \qquad (f, g \in L^1(\mathbb{R}^+),\ \text{a.e. } t \in \mathbb{R}^+),$$

and $\mathscr{L}(X)$ is the algebra of all bounded linear operators on X. The homomorphism H can be viewed as an integrated form of the semigroup $\{\mathscr{S}(t)\}_{t\geq 0}$ (see, e.g., [95, Proposition 2.23] in regard to this terminology). We hasten to remark that not every continuous homomorphism from $L^1(\mathbb{R}^+)$ into $\mathscr{L}(X)$ is of the form as above. Similarly, uniformly bounded, strongly continuous one-parameter groups and cosine families of operators lead to continuous homomorphisms from $L^1(\mathbb{R})$ and from $L^1_{\mathrm{e}}(\mathbb{R})$, respectively. Here $L^1(\mathbb{R})$ is the Banach algebra of equivalence classes of complex-valued, Lebesgue integrable functions on the real line $\mathbb{R}$, and $L^1_{\mathrm{e}}(\mathbb{R})$ is the Banach algebra of equivalence classes of complex-valued, Lebesgue integrable, *even* functions on $\mathbb{R}$, these algebras being equipped with the convolution product

$$(f \star g)(t) = \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s \qquad (f, g \in L^1(\mathbb{R}) \text{ or } f, g \in L^1_{\mathrm{e}}(\mathbb{R}), \text{ a.e. } t \in \mathbb{R}^+).$$

Again, the homomorphisms from $L^1(\mathbb{R})$ and from $L^1_{\mathrm{e}}(\mathbb{R})$ induced by uniformly bounded one-parameter groups or cosine families can be treated as integrated forms of the underlying operator groups or cosine families, and, as in the case of homomorphisms from $L^1(\mathbb{R}^+)$, there are more continuous homomorphisms from $L^1(\mathbb{R})$ and from $L^1_{\mathrm{e}}(\mathbb{R})$ than just the integrated forms of uniformly bounded one-parameter operator groups or cosine families.

In this memoir, we shall study isolated points of spaces of homomorphisms from various Banach algebras that include, as particular cases, the three algebras just mentioned: $L^1(\mathbb{R}^+)$, $L^1(\mathbb{R})$, and $L^1_{\mathrm{e}}(\mathbb{R})$. The latter algebras are representative of three classes of algebras that will serve as domains of homomorphisms, namely the semigroup algebras of certain semigroups of locally compact groups, the group algebras of locally compact Abelian groups, and the convolution algebras of (equivalence classes of) Haar integrable, even functions on locally compact Abelian groups. Continuous homomorphisms from algebras in these classes are closely linked to operator-valued semigroups, groups, and cosine families indexed by either semigroups or groups that include $\mathbb{R}^+$ and $\mathbb{R}$ as particular cases. The relevant links will turn out to be instrumental to our analysis.

The semigroup algebras, the group algebras, and the convolution algebras of integrable, even functions on groups considered in the memoir will, as it happens, be members of a certain common class of complex Banach algebras that we shall refer to as the ordered AL-algebras. An ordered AL-algebra is a complex AL-space—a space which is a special case of a complex Banach lattice—and simultaneously a Banach algebra, with a multiplication tied to the order structure in a specific fashion. The link between the multiplication and the order is captured by the requirement that the positive cone associated with the underlying partial ordering be closed under multiplication, and that the norm restricted to the positive cone be multiplicative.

The reason behind singling out the ordered AL-algebras as a particular point of interest is that every non-zero algebra in this class admits a special linear multiplicative functional of norm 1, termed here the fundamental character on the algebra. Informally speaking, in many cases where the ordered AL-algebra consists of complex-valued functions on a given set, the fundamental character is represented as an integral over that set. Any fundamental character naturally gives rise to isolated homomorphisms in appropriate spaces of homomorphisms. More specifically, if $\mathfrak{L}$ is a non-zero, complex ordered AL-algebra with fundamental character l, and if $\mathfrak{A}$ is a unital normed algebra with identity $e_{\mathfrak{A}}$, then the associated homomorphism $e_{\mathfrak{A}} \otimes l \colon \mathfrak{L} \to \mathfrak{A}$ defined by

$$(e_{\mathfrak{A}} \otimes l)x = l(x)e_{\mathfrak{A}} \qquad (x \in \mathfrak{L})$$

has the following property: given a continuous homomorphism $H \colon \mathfrak{L} \to \mathfrak{A}$, and this homomorphism may be non-unital while $\mathfrak{L}$ is unital, the condition

$$\|H - e_{\mathfrak{A}} \otimes l\| < 1 \tag{1.1}$$

implies that $H = e_{\mathfrak{A}} \otimes l$. This, in particular, means that $e_{\mathfrak{A}} \otimes l$—which hereafter will be referred to as the fundamental homomorphism from $\mathfrak{L}$ to $\mathfrak{A}$—is an isolated homomorphism among all continuous homomorphisms from $\mathfrak{L}$ to $\mathfrak{A}$. With the existence of isolated homomorphisms thus ensured, a natural question arises as to whether there exist isolated homomorphisms from ordered AL-algebras that are different from fundamental homomorphisms. As we establish in this memoir, the answer is in the affirmative for each of the algebras $L^1(\mathbb{R}^+)$, $L^1(\mathbb{R})$, and $L^1_{\mathrm{e}}(\mathbb{R})$, and, generally, for the classes of algebras for which $L^1(\mathbb{R}^+)$, $L^1(\mathbb{R})$, and $L^1_{\mathrm{e}}(\mathbb{R})$ are representative members.

The constant 1 in condition (1.1) cannot in general be replaced by a greater number. Indeed, if $\mathfrak{L}$ is a non-zero ordered AL-algebra with fundamental character l, and if $\mathfrak{A}$ is a unital normed algebra, then, letting 0 denote the zero homomorphism from $\mathfrak{L}$ to $\mathfrak{A}$, we have

$$\|0 - e_{\mathfrak{A}} \otimes l\| = \|e_{\mathfrak{A}} \otimes l\| = \|e_{\mathfrak{A}}\|\|l\| = 1,$$

so $\|0 - e_{\mathfrak{A}} \otimes l\| < \alpha$ for every $\alpha > 1$, and yet $0 \neq e_{\mathfrak{A}} \otimes l$. Moreover, if $\mathfrak{A}$ possesses a non-trivial idempotent e ($0 \neq e \neq e_{\mathfrak{A}}$) of norm 1, then, with $(e_{\mathfrak{A}} - e) \otimes l$ denoting the homomorphism from $\mathfrak{L}$ to $\mathfrak{A}$ defined by

$$((e_{\mathfrak{A}} - e) \otimes l)(x) = l(x)(e_{\mathfrak{A}} - e) \quad (x \in \mathfrak{L}),$$

we have

$$\|(e_{\mathfrak{A}} - e) \otimes l - e_{\mathfrak{A}} \otimes l\| = \|e\|\,\|l\| = 1,$$

so $\|(e_{\mathfrak{A}} - e) \otimes l - e_{\mathfrak{A}} \otimes l\| < \alpha$ for every $\alpha > 1$, but clearly $(e_{\mathfrak{A}} - e) \otimes l \neq e_{\mathfrak{A}} \otimes l$. Having said that, if $\mathfrak{A}$ is a unital normed algebra *without non-trivial idempotents*, then it may well be that there exists $\alpha > 1$ such that, if H is a *non-zero*, continuous homomorphism from $\mathfrak{L}$ to $\mathfrak{A}$ satisfying

$$\|H - e_{\mathfrak{A}} \otimes l\| < \alpha,$$

then $H = e_{\mathfrak{A}} \otimes l$. This leads to the following problem: given a non-zero ordered AL-algebra $\mathfrak{L}$, find the greatest positive number $\alpha(\mathfrak{L})$ such that, for every non-zero, continuous homomorphism H from $\mathfrak{L}$ to a unital normed algebra $\mathfrak{A}$ without non-trivial idempotents, the condition

$$\|H - e_{\mathfrak{A}} \otimes l\| < \alpha(\mathfrak{L})$$

implies that $H = e_{\mathfrak{A}} \otimes l$. Calling $\alpha(\mathfrak{L})$ so defined the α-number of $\mathfrak{L}$, we take as one of our goals to calculate α-numbers for various ordered AL-algebras. The α-numbers turn out to be invariant under isometric isomorphisms of ordered algebras, and this can be exploited for distinguishing between isometrically non-isomorphic ordered AL-algebras. For example, by virtue of having different α-numbers, the algebras $L^1(\mathbb{R}^+)$ and $L^1(\mathbb{R})$ and the algebras $L^1(\mathbb{R}^+)$ and $L^1_{\mathrm{e}}(\mathbb{R})$ turn out not to be isometrically algebra and order isomorphic.

Having indicated the gist of our findings, we now proceed to overview individual chapters of the memoir.

Chapter 2 introduces ordered AL-algebras and related entities—fundamental characters and fundamental homomorphisms. In the same chapter, the critical isolability property of the fundamental homomorphisms from ordered AL-algebras—one that has already been mentioned above—is established. Following that, the α-number of an ordered AL-

algebra is introduced, and its invariance to isometric algebra and order isomorphisms is proved. Moreover, various notions of isolability and accessibility of homomorphisms, including a strong form of isolability which we term total isolability, are introduced and discussed. It is proved that every totally isolated homomorphism takes a particularly simple form—it is scalar in a specific sense.

In Chapter 3 we consider the semigroup algebras of certain subsemigroups of locally compact groups and their homomorphisms. We single out a vast class of semigroup algebras for which the α-number is equal to 1. We also reveal totally isolated homomorphisms from semigroup algebras that are different from fundamental homomorphisms.

Chapter 4 concerns three concrete semigroup algebras, one of which is $L^1(\mathbb{R}^+)$. The specific nature of the underlying semigroups allows us to discuss comprehensively isolability properties of homomorphisms from the algebras in question. In particular, we show that $\alpha(L^1(\mathbb{R}^+)) = 1$, and that some scalar homomorphisms from $L^1(\mathbb{R}^+)$ are isolated, and some are not.

Chapter 5 concerns the group algebras of locally compact Abelian groups and their homomorphisms. We show that in many cases the calculation of the α-number of a particular group algebra can be reduced to the calculation of a certain numerical characteristic of the underlying group, which we term the β-number of the group. Using purely harmonic-analytic arguments, we compute β-numbers for many groups, including all finite Abelian groups. This leads to the computation of the α-numbers of many group algebras. In particular, we find that $\alpha(L^1(\mathbb{R})) = 2$. We demonstrate that generally the α-numbers of group algebras are no smaller than $\sqrt{3}$. Moreover, all scalar homomorphisms from group algebras are shown to be totally isolated.

Chapter 6 concerns the convolution algebras of Haar integrable, even functions on locally compact Abelian groups and their homomorphisms. We introduce a numerical characteristic of a pair comprising a locally compact Abelian group and a bounded, scalar cosine function on that group; we call this characteristic the γ-number of the pair. We use γ-numbers to calculate the α-numbers of the convolution algebras of integrable, even functions on various groups. We find, among other things, that $\alpha(L^1_{\mathrm{e}}(\mathbb{R})) = 2$. We establish that generally the α-numbers of convolution algebras of integrable, even functions are no smaller than $\sqrt{5}/2$. We also show that, as in the case of scalar homomorphisms from group algebras, all scalar homomorphisms from convolution algebras of integrable, even functions are totally isolated. As an independent result, one of two similar ones, we prove that $L^1(\mathbb{R})$ and $L^1_{\mathrm{e}}(\mathbb{R})$ are not isometrically isomorphic as ordered algebras. Given that $\alpha(L^1(\mathbb{R})) = \alpha(L^1_{\mathrm{e}}(\mathbb{R})) = 2$, proving this fact requires an invariant different from the α-number. The invariant that we devise allows us to establish that $L^1(\mathbb{R})$ and $L^1_{\mathrm{e}}(\mathbb{R})$ not only are not isometrically isomorphic as ordered algebras, but in fact are not isometrically isomorphic as normed algebras.

Finally, the short seventh chapter furnishes a list of nine ordered AL-algebras, no two of which are isometrically algebra and order isomorphic. This provides a kind of symbolic summary of the developments reported in the memoir.

Basic notation. We shall use the following notation: $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers; $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the set of integers; $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ is the set of non-negative integers; $\mathbb{Q}$ is the set of rational numbers; $\mathbb{C}$ is the complex plane; $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in the complex plane; $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disc in the complex plane; $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the closed unit disc in the complex plane; $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ is the open complex right half-plane; $\overline{\mathbb{C}^+} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$ is the closed complex right half-plane; $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$ is the closed complex upper half-plane; $\mathrm{i}\mathbb{R} = \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\}$ is the vertical line in the complex plane passing through the plane's origin.

For $n \in \mathbb{N}$, we set

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\};$$

this set is a group with respect to addition modulo n, the additive group of integers modulo n. Further, we set

$$\mathbb{U}_n = \left\{ \exp\left(\frac{2\pi \mathrm{i} k}{n}\right) \,\middle|\, k = 0, \ldots, n-1 \right\};$$

this set is a group under multiplication, the multiplicative group of nth roots of unity.

The cardinality of a set A is denoted by $|A|$.

The characteristic function of a subset A of a set is denoted by χ_A. The function constantly equal to 1 on a set A is denoted by 1_A.

2. Ordered AL-algebras

2.1. Definitions. As already foreshadowed in the Introduction, fundamental to our investigations will be a special class of complex Banach lattices, each member of which is an AL-space equipped with a multiplication which gives the space the structure of a Banach algebra and satisfies certain additional conditions. We precede the definition of that class with some concepts and results.

We first recall the notion of a *complex Banach lattice*. As a linear space, a complex Banach lattice is the complexification of a real Banach lattice. Generally, the complexification of a real Banach space can be endowed with various norms, each extending the norm of the original real space [62], [86]. In contrast, a complex Banach lattice comes, by definition, equipped with a specific norm which extends the norm of the underlying real Banach lattice. Going into more detail, suppose that X is the complexification of a real Banach lattice $(X_{\mathbb{R}}, \|\cdot\|)$ with a partial ordering $\leq$. Given $x \in X_{\mathbb{R}}$, let

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad \text{and} \qquad |x| = x \vee (-x).$$

Let $X^+ = X^+_{\mathbb{R}} = \{x \in X_{\mathbb{R}} \mid x \geq 0\}$. The set X^+ is called the *positive cone* of X. Consider $x \in X$ written uniquely as $x = u + \mathrm{i}v$, where $u, v \in X_{\mathbb{R}}$, and define the *modulus* $|x| \in X^+$ of x by

$$|x| = \bigvee \{u \cos\theta + v \sin\theta \mid 0 \leq \theta < 2\pi\}.$$

That the supremum in X^+ here exists is a non-trivial fact—see [1, p. 104] or [59, Prop. 2.2.1]. Alternatively, one may define the modulus of x by

$$|x| = (|u|^2 + |v|^2)^{1/2},$$

where the expression on the right-hand side is formed with the aid of the functional calculus of Youdine [98] and Krivine [50]. As explained in [26, pp. 326–329] and [54, pp. 40–42], if $y_1, \dots, y_n$ are elements of a real Banach lattice Y and if $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous and positively homogeneous of degree 1, then one can always define $f(y_1, \dots, y_n) \in Y$. The mapping $f \mapsto f(y_1, \dots, y_n)$ is linear and preserves the lattice operations whenever $y_1, \dots, y_n \in Y^+$; in particular, if $f(s_1, \dots, s_n) \geq 0$ whenever $s_1, \dots, s_n \geq 0$, then $f(y_1, \dots, y_n) \geq 0$ whenever $y_1, \dots, y_n \in Y^+$. In line with this, the second definition of the modulus of x results from applying the Youdine–Krivine calculus to the function $f(s_1, s_2) = (|s_1|^2 + |s_2|^2)^{1/2}$.

It can be shown that, for $\alpha \in \mathbb{C}$ and $x, y \in X$, we have: $|x| = 0$ if and only if $x = 0$; $|\alpha x| = |\alpha|\,|x|$; $|x + y| \leq |x| + |y|$. These properties ensure that setting

$$\|x\| = \||x|\| \qquad (x \in X)$$

defines a norm on X. Moreover, given $x = u + \mathrm{i}v \in X$, the inequalities $|u| \leq |x|$, $|v| \leq |x|$, and $|x| \leq |u| + |v|$ imply that

$$\tfrac{1}{2}(\|u\| + \|v\|) \leq \|x\| \leq \|u\| + \|v\|.$$

Therefore, $\|\cdot\|$ not only is a norm on X, but is in fact a norm equivalent to the standard norm on X defined by

$$\|x\|_{\mathbb{C}} = \sup_{0 \leq \theta < 2\pi} \|u \cos\theta + v \sin\theta\| \qquad (x \in X).$$

The Banach space $(X, \|\cdot\|)$ is precisely what one means by the complex Banach lattice with underlying real Banach lattice $X_{\mathbb{R}}$. It is clear that the norm of X extends the norm of $X_{\mathbb{R}}$ and, moreover, has the property that, if $x, y \in X$ are such that $|x| \leq |y|$, then $\|x\| \leq \|y\|$.

If X is a complex Banach lattice, then the underlying real Banach lattice $X_{\mathbb{R}}$ is called the real part of X and is uniquely determined as the real linear span of X^+.

For details of the above remarks, see [1, §3.2], [74, Chapter II, §11], [85], or [99].

We next recall the notion of an AL-space. A (real or complex) Banach lattice $(X, \|\cdot\|)$ is an *AL-space* (or abstract L-space) if

$$\|x + y\| = \|x\| + \|y\| \quad \text{whenever } x, y \in X^+ \text{ with } x \wedge y = 0;$$

see [4, Def. 4.20], [54, Def. 1.b.1], or [59, Def. 1.4.6]. An equivalent definition, due to S. Kakutani [44], states that a Banach lattice X is an AL-space if and only if

$$\|x + y\| = \|x\| + \|y\| \quad \text{whenever } x, y \in X^+.$$

For an argument establishing the equivalence of the two definitions, see [51, §2, Theorem 6] or [75, p. 99]. The equivalence also follows easily from the representation theorem for AL-spaces conforming to the first definition—see [4, p. 200] or [54, Remark 1, p. 17]. It is a standard fact that every space of the form $L^1(\Omega, \mu)$, where (Ω, μ) is a measure space, is an AL-space.

We are now ready to introduce a definition.

DEFINITION 2.1. Let $\mathfrak{L}$ be a complex Banach algebra. Then:

(i) $\mathfrak{L}$ is an *AL-algebra* if there is a partial order $\leq$ in terms of which $\mathfrak{L}$ is a complex AL-space;
(ii) $\mathfrak{L}$ is an *ordered AL-algebra* if $\mathfrak{L}$ is an AL-algebra such that

$$xy \in \mathfrak{L}^+ \quad \text{and} \quad \|xy\| = \|x\|\,\|y\| \quad \text{whenever } x, y \in \mathfrak{L}^+. \tag{2.1}$$

REMARKS. (1) The terminology of the above definition is a minor modification of the terminology introduced by White [90].

(2) An AL-algebra may fail to be an ordered AL-algebra. For example, as pointed out in [90], the space $\ell^1(\mathbb{Z})$ of all complex-valued, summable functions on $\mathbb{Z}$, taken with coordinatewise multiplication and the usual order, is an AL-algebra, but not an ordered AL-algebra.

(3) There are many examples of ordered AL-algebras, among them various convolution algebras such as semigroup algebras, group algebras, and convolution algebras of integrable, even functions on groups. Specific algebras representing each of the three last-mentioned categories will emerge later on and will play a vital role in the ensuing study.

(4) Every ordered AL-algebra is a *Banach lattice algebra*. A Banach lattice algebra is a Banach algebra and simultaneously a Banach lattice such that the positive cone associated with the underlying partial ordering is closed under multiplication [23], [91], [92]. Of the two types of algebras, Banach lattice algebras and ordered AL-algebras, the latter will be more adequate for our considerations. Basic general arguments to be put forward in what follows require that the norm restricted to the positive cone be both additive and multiplicative, and as such are geared to work for ordered AL-algebras rather than for general Banach lattice algebras.

2.2. The fundamental character. Given a complex ordered AL-algebra $\mathfrak{L}$ with real part $\mathfrak{L}_{\mathbb{R}}$, we define a mapping $l\colon \mathfrak{L}_{\mathbb{R}} \to \mathbb{R}$ by

$$l(x) = \|x^+\| - \|x^-\| \qquad (x \in \mathfrak{L}_{\mathbb{R}}).$$

Note that $l(x) = \|x\|$ for $x \in \mathfrak{L}^+$, so, in particular, $l(x) \geq 0$ whenever $x \in \mathfrak{L}^+$. Other basic properties of l are summarised in the lemma that follows. That lemma draws upon a fundamental extension theorem due to Kantorovich [47]; see also, e.g., [4, Theorem 1.10], [3, Lemma 8.23], or [3, Theorem 9.30].

LEMMA 2.2. *The mapping l has the following properties:*

(i) *if $x \in \mathfrak{L}_{\mathbb{R}}$ is represented as $x = x_1 - x_2$ with $x_1, x_2 \in \mathfrak{L}_{\mathbb{R}}^+$, then $l(x) = l(x_1) - l(x_2)$;*
(ii) *l is linear;*
(iii) *l is multiplicative.*

Proof. (i) Since $x_1 \geq 0$ and $x_1 = x + x_2 \geq x$, we see that $x_1 \geq x^+$. Similarly, $x_2 \geq x^-$. Now, introducing $h := x_2 - x^- = x_1 - x^+$, we have

$$\begin{aligned} l(x_1) - l(x_2) &= \|x^+ + h\| - \|x^- + h\| = \|x^+\| + \|h\| - \|x^-\| - \|h\| \\ &= \|x^+\| - \|x^-\| = l(x). \end{aligned}$$

(ii) Homogeneity is established as follows: if $\alpha \geq 0$ and if $x \in \mathfrak{L}_{\mathbb{R}}$, then αx^+ and αx^- are non-negative, and, by (i), $l(\alpha x) = \|\alpha x^+\| - \|\alpha x^-\| = \alpha(\|x^+\| - \|x^-\|) = \alpha l(x)$; the case $\alpha < 0$ is treated similarly. Additivity is also a consequence of (i): if $x, y \in \mathfrak{L}_{\mathbb{R}}$, then

$$\begin{aligned} l(x+y) &= l(x^+ + y^+ - x^- - y^-) = \|x^+ + y^+\| - \|x^- + y^-\| \\ &= \|x^+\| + \|y^+\| - \|x^-\| - \|y^-\| = l(x) + l(y). \end{aligned}$$

(iii) If $x, y \in \mathfrak{L}_{\mathbb{R}}$, then, by (2.1),

$$\begin{aligned} l(xy) &= l\big((x^+ - x^-)(y^+ - y^-)\big) = l\Big(\underbrace{x^+y^+}_{\geq 0} + \underbrace{x^-y^-}_{\geq 0} - \underbrace{x^+y^-}_{\geq 0} - \underbrace{x^-y^+}_{\geq 0}\Big) \\ &= \|x^+y^+\| + \|x^-y^-\| - \|x^+y^-\| - \|x^-y^+\| \\ &= \|x^+\|\,\|y^+\| + \|x^-\|\,\|y^-\| - \|x^+\|\,\|y^-\| - \|x^-\|\,\|y^+\| \\ &= (\|x^+\| - \|x^-\|)(\|y^+\| - \|y^-\|) = l(x)l(y). \ \blacksquare \end{aligned}$$

As any other real linear functional, l can be uniquely extended to a complex linear functional on $\mathfrak{L}$. Denoting the extended functional again by l, we have

$$l(x) = l(u) + \mathrm{i}l(v) \qquad (x = u + \mathrm{i}v \in \mathfrak{L},\ u, v \in \mathfrak{L}_{\mathbb{R}}).$$

It is straightforward to verify that the complex-valued functional l is complex multiplicative.

Proposition 2.3. *If $\mathfrak{L}$ is a non-zero ordered AL-algebra and if l is the complex-valued functional on $\mathfrak{L}$ introduced above, then $\|l\| = 1$.*

Proof. Since l is linear multiplicative, we have $\|l\| \leq 1$ (cf. [15, §16, Proposition 3]), and since $l(x) = \|x\|$ for every $x \in \mathfrak{L}^+$ and $\mathfrak{L}^+$ is non-zero, we in fact have $\|l\| = 1$. $\blacksquare$

Convention. All ordered AL-spaces considered from now on will be tacitly assumed to be non-zero.

It is customary to refer to a non-zero, complex-valued linear multiplicative functional on a complex algebra as a *character* on that algebra. Given a complex algebra $\mathfrak{A}$, the collection of all characters on $\mathfrak{A}$ will be denoted by $\Delta(\mathfrak{A})$, and called the *character space* of $\mathfrak{A}$.

Definition 2.4. Let $\mathfrak{L}$ be an ordered AL-algebra. Then the corresponding functional l is the *fundamental character* on $\mathfrak{L}$.

We record the following immediate consequence of Proposition 2.3:

Proposition 2.5. *For every ordered AL-algebra $\mathfrak{L}$, $\Delta(\mathfrak{L})$ is non-empty.*

We digress momentarily to mention that a non-commutative Banach algebra may have an empty character space. A well-known example of such an algebra is $\mathscr{L}(H)$, where H is a Hilbert space of dimension at least 2 (that this is the case can be inferred, e.g., from the following two facts:

(i) $\mathscr{L}(H)$ is simple when H is finite-dimensional [10, Sect. 13.6, Theorem 10];
(ii) every operator in $\mathscr{L}(H)$ is the sum of two commutators when H is infinite-dimensional; see [37, Corollary to Theorem 8] or [96, Theorem 6.1]).

In connection with this example, it is interesting to note that there exist infinite-dimensional Banach spaces X, none of which is isomorphic to a Hilbert space, such that the algebra $\mathscr{L}(X)$ admits a character [58], [60], [78], [93], [94].

2.3. Fundamental homomorphisms. We now reveal homomorphisms from ordered AL-algebras that have a distinctive isolability property. The development in this section is central to the rest of the memoir.

Given two normed spaces X and Y, we denote by $\mathscr{L}(X, Y)$ the space of all bounded linear operators from X to Y, endowed with the corresponding operator norm.

Suppose that X is a normed space. The dual space of X is denoted by X^*. The value of a functional $x^* \in X^*$ at $x \in X$ is written $x^*(x)$. We abbreviate $\mathscr{L}(X, X)$ as $\mathscr{L}(X)$. The identity operator on X is denoted by I_X. We recall for the record that, if X is non-zero, then $\mathscr{L}(X)$ is a unital normed algebra, with I_X as the identity element; the non-nullity of X is required to ensure that $\|I_X\| = 1$, in line with the standard convention whereby the identity of a unital normed algebra has norm 1.

Suppose that X and Y are two normed spaces. For $x^* \in X^*$ and $y \in Y$, let $y \otimes x^*$ denote the operator in $\mathscr{L}(X, Y)$ given by

$$(y \otimes x^*)(x) = x^*(x)y \quad (x \in X).$$

It is readily seen that $\|y \otimes x^*\| = \|x^*\| \, \|y\|$.

Let $\mathfrak{X}$ and $\mathfrak{A}$ be two normed algebras. We denote by $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ the set of all continuous algebra homomorphisms from $\mathfrak{X}$ to $\mathfrak{A}$. It is immediate that, if e is an idempotent in $\mathfrak{A}$ and if ϕ is a character on $\mathfrak{X}$, then $e \otimes \phi$ is a homomorphism in $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$. In particular, if $\mathfrak{A}$ is unital, then, for each $\phi \in \Delta(\mathfrak{X})$, $e_{\mathfrak{A}} \otimes \phi$ is a homomorphism in $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$.

Definition 2.6. Let $\mathfrak{L}$ be an ordered AL-algebra with fundamental character l, and let $\mathfrak{A}$ be a unital normed algebra. Then $e_{\mathfrak{A}} \otimes l$ is the *fundamental homomorphism* from $\mathfrak{L}$ to $\mathfrak{A}$.

The pertinence of introducing the fundamental homomorphisms lies in the following result.

Theorem 2.7. *Let $\mathfrak{L}$ be an ordered AL-algebra with fundamental character l, and let $\mathfrak{A}$ be a unital normed algebra. If $H \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})$ is such that $\|H - e_{\mathfrak{A}} \otimes l\| < 1$, then $H = e_{\mathfrak{A}} \otimes l$. Moreover, the above statement fails in general if '$<$' is replaced by '$\leq$', namely:*

(i) *it fails for $H = 0$;*
(ii) *it does not hold in general for non-zero $H \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})$ because of the following fact: if $\mathfrak{A}$ is a unital normed algebra with a non-trivial idempotent of norm 1, then there exists a non-zero $H \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})$ such that $\|H - e_{\mathfrak{A}} \otimes l\| = 1$.*

Remark 2.8. There is an abundance of unital normed algebras having non-trivial idempotents of norm 1. The following observation will be of relevance later on: if X is a normed space of dimension at least 2, then $\mathscr{L}(X)$ possesses a non-trivial idempotent of norm 1. Indeed, if $x \in X$ is such that $\|x\| = 1$, and if $x^* \in X^*$ is such that $\|x^*\| = 1$ and $x^*(x) = \|x\|$, then $x \otimes x^*$ is an idempotent in $\mathscr{L}(X)$ such that $\|x \otimes x^*\| = \|x\| \, \|x^*\| = 1$.

Moreover, $x \otimes x^*$ is non-trivial: it is clearly non-zero, and since its range is one-dimensional, and hence has dimension smaller than the dimension of X, it also differs from I_X.

The proof of Theorem 2.7 will be based on the following result due to Cox [20], Nakamura and Yoshida [64], Hirschfeld [42], and Wallen [89]:

PROPOSITION 2.9. *If $\mathfrak{A}$ is a unital normed algebra and if $x \in \mathfrak{A}$ with*

$$\sup_{n \in \mathbb{N}} \|x^n - e_{\mathfrak{A}}\| < 1,$$

then $x = e_{\mathfrak{A}}$.

It is worth mentioning that a slightly stronger version of the above proposition holds true, and that this stronger version can be proved by a remarkably short argument; see [89].

Proof of Theorem 2.7. For the first part, given that every element of $\mathfrak{L}$ is a linear combination of elements of $\mathfrak{L}^+$, we need only show that $H(x) = (e_{\mathfrak{A}} \otimes l)(x)$ whenever $x \in \mathfrak{L}^+$ and $\|x\| = 1$. Let $x \in \mathfrak{L}^+$ be such that $\|x\| = 1$. Then, for every $n \in \mathbb{N}$, $l(x^n) = \|x^n\| = \|x\|^n = 1$, and further

$$\|(H(x))^n - e_{\mathfrak{A}}\| = \|H(x^n) - (e_{\mathfrak{A}} \otimes l)(x^n)\| \leq \|H - e_{\mathfrak{A}} \otimes l\| \, \|x^n\| = \|H - e_{\mathfrak{A}} \otimes l\|.$$

Since $\|H - e_{\mathfrak{A}} \otimes l\| < 1$ by assumption, we see that $\sup_{n \in \mathbb{N}} \|((H(x))^n - e_{\mathfrak{A}}\| < 1$. An application of Proposition 2.9 now implies that $H(x) = e_{\mathfrak{A}}$. Taking into account that $e_{\mathfrak{A}} = l(x) e_{\mathfrak{A}} = (e_{\mathfrak{A}} \otimes l)(x)$, we conclude that $H(x) = (e_{\mathfrak{A}} \otimes l)(x)$, as required.

For the second part, suppose first that $\mathfrak{A}$ is an arbitrary unital normed algebra. With 0 denoting the zero homomorphism from $\mathfrak{L}$ to $\mathfrak{A}$, we have

$$\|0 - e_{\mathfrak{A}} \otimes l\| = \|e_{\mathfrak{A}} \otimes l\| = \|e_{\mathfrak{A}}\| \, \|l\| = 1,$$

and then also $0 \neq e_{\mathfrak{A}} \otimes l$. This proves assertion (i).

Suppose now that $\mathfrak{A}$ is a unital normed algebra with a non-trivial idempotent e of norm 1. Then $f = e_{\mathfrak{A}} - e$ is a non-trivial idempotent and $f \otimes l$ is a non-zero homomorphism in $\mathrm{Hom}(\mathfrak{L}, \mathfrak{A})$. Moreover,

$$\|f \otimes l - e_{\mathfrak{A}} \otimes l\| = \|e_{\mathfrak{A}}\| \, \|l\| = 1.$$

The internal statement in assertion (ii) follows, and thereby so do the assertion itself and the entire theorem. ∎

2.4. The α-number. A unital algebra $\mathfrak{A}$ is said to have no non-trivial idempotents if the only idempotents of $\mathfrak{A}$ are the zero element and the identity element of $\mathfrak{A}$. We denote by $\mathscr{A}_{\mathrm{wni}}$ the class of complex, unital normed algebras without non-trivial idempotents. This class is obviously non-void. For example, if S is a connected topological space, then the collection $C_{\mathrm{b}}(S)$ of all bounded, continuous, complex-valued functions on S, taken with pointwise operations of addition, scalar multiplication, and product, and equipped with the uniform norm, is a unital Banach algebra without non-trivial idempotents.

Let $\mathfrak{L}$ be an ordered AL-algebra with fundamental character l. If $\mathfrak{A}$ is a unital normed algebra, then, by Theorem 2.7, the infimum of $\|H - e_{\mathfrak{A}} \otimes l\|$ over all $H \in \mathrm{Hom}(\mathfrak{L}, \mathfrak{A})$ different from $e_{\mathfrak{A}} \otimes l$ is no smaller than 1; and if only $\mathfrak{A}$ has no non-trivial idempotent and the zero homomorphism is excluded from the count, chances are that the infimum will be strictly greater than 1. This motivates the following definition.

DEFINITION 2.10. Let $\mathfrak{L}$ be an ordered AL-algebra with fundamental character l. For a unital normed algebra $\mathfrak{A}$, let $\mathrm{Hom}_\bullet(\mathfrak{L},\mathfrak{A})$ denote the set of all non-zero, continuous homomorphisms from $\mathfrak{L}$ to $\mathfrak{A}$ different from $e_{\mathfrak{A}} \otimes l$. Then

$$\alpha(\mathfrak{L}) := \inf_{\substack{\mathfrak{A}\in\mathscr{A}_{\mathrm{wni}} \\ H\in\mathrm{Hom}_\bullet(\mathfrak{L},\mathfrak{A})}} \|H - e_{\mathfrak{A}} \otimes l\|$$

is the *α-number* of $\mathfrak{L}$.

To proceed further, we require one more definition.

DEFINITION 2.11. Let $(\mathfrak{L}_1, \|\cdot\|_{\mathfrak{L}_1})$ and $(\mathfrak{L}_2, \|\cdot\|_{\mathfrak{L}_2})$ be two ordered AL-algebras. A linear map $I\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ is an *isometric isomorphism of ordered algebras* (or *isometric algebra and order isomorphism*) if the following conditions are satisfied:

(i) I is an isometric isomorphism of Banach spaces, i.e., $\|I(x)\|_{\mathfrak{L}_2} = \|x\|_{\mathfrak{L}_1}$ for $x \in \mathfrak{L}_1$ and I is 'onto';
(ii) I is a homomorphism of Banach algebras, i.e., $I(xy) = I(x)I(y)$ for $x, y \in \mathfrak{L}_1$;
(iii) I preserves the order, i.e., $I(x) \in \mathfrak{L}_2^+$ whenever $x \in \mathfrak{L}_1^+$.

THEOREM 2.12. *The α-number is invariant under isometric isomorphisms of ordered algebras.*

Proof. Let $(\mathfrak{L}_1, \|\cdot\|_{\mathfrak{L}_1})$ and $(\mathfrak{L}_2, \|\cdot\|_{\mathfrak{L}_2})$ be two ordered AL-algebras with fundamental characters l_1 and l_2, respectively. Let $I\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ be an isometric isomorphism of ordered algebras. If $x \in \mathfrak{L}_1^+$, then $I(x) \in \mathfrak{L}_2^+$, and so $l_2(I(x)) = \|I(x)\|_{\mathfrak{L}_2} = \|x\|_{\mathfrak{L}_1} = l_1(x)$. Since every element of $\mathfrak{L}_1$ is a linear combination of elements of $\mathfrak{L}_1^+$, we have $l_2(I(x)) = l_1(x)$ for all $x \in \mathfrak{L}_1$. In other words, $l_1 = l_2 \circ I$. It is now clear that, if $\mathfrak{A}$ is a unital normed algebra, then

$$e_{\mathfrak{A}} \otimes l_1 = (e_{\mathfrak{A}} \otimes l_2) \circ I.$$

Suppose that $\mathfrak{A} \in \mathscr{A}_{\mathrm{wni}}$. Let H be a non-zero homomorphism in $\mathrm{Hom}(\mathfrak{L}_2, \mathfrak{A})$ such that $\|H - e_{\mathfrak{A}} \otimes l_2\| < \alpha(\mathfrak{L}_1)$. Then $G := H \circ I$ is a non-zero homomorphism in $\mathrm{Hom}(\mathfrak{L}_1, \mathfrak{A})$ and

$$\begin{aligned}\|G - e_{\mathfrak{A}} \otimes l_1\| &= \sup_{\|x\|_{\mathfrak{L}_1}=1} \|H(I(x)) - (e_{\mathfrak{A}} \otimes l_2)(I(x))\| = \sup_{\|x\|_{\mathfrak{L}_2}=1} \|H(x) - (e_{\mathfrak{A}} \otimes l_2)(x)\| \\ &= \|H - e_{\mathfrak{A}} \otimes l_2\| < \alpha(\mathfrak{L}_1).\end{aligned}$$

It now follows, by definition of $\alpha(\mathfrak{L}_1)$, that $G = e_{\mathfrak{A}} \otimes l_1$. Therefore, $H = (e_{\mathfrak{A}} \otimes l_1) \circ I^{-1} = e_{\mathfrak{A}} \otimes l_2$, showing that $\alpha(\mathfrak{L}_2) \geq \alpha(\mathfrak{L}_1)$. As the roles of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ can be interchanged, we conclude that $\alpha(\mathfrak{L}_1) = \alpha(\mathfrak{L}_2)$. ■

As it turns out, α-numbers can be calculated in many interesting cases and can be effectively used to distinguish between isometrically non-isomorphic ordered AL-algebras. We shall elaborate on this shortly, but first we shall discuss different, though related, matters.

2.5. Isolability and accessibility of homomorphisms. Theorem 2.7 makes it evident that fundamental homomorphisms from ordered AL-algebras are isolated. What it means for a homomorphism to be isolated is rather clear, nonetheless Definition 2.13 below gives a clear-cut statement. As it turns out, fundamental homomorphisms from ordered AL-algebra enjoy a slightly stronger property than just mere isolability. We term this property *total isolability*; its precise formulation is also given in Definition 2.13. In the present section, we start a systematic study of the question of recognising which homomorphisms from ordered AL-algebras are totally isolated. Some of the results obtained here will be used in subsequent sections to reveal totally isolated homomorphisms that are different from fundamental homomorphisms.

If $\mathfrak{X}$ and $\mathfrak{A}$ are two normed algebras, then $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ is part of $\mathscr{L}(\mathfrak{X}, \mathfrak{A})$, and this, in particular, implies that the distance between members of $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ can be measured in the operator norm of $\mathscr{L}(\mathfrak{X}, \mathfrak{A})$.

DEFINITION 2.13. Let $\mathfrak{X}$ be a normed algebra, let $\mathfrak{A}$ be a unital normed algebra, and let $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$. Then:

- (i) H is *scalar* if, for all $x \in \mathfrak{X}$, $H(x)$ is a scalar multiple of $e_{\mathfrak{A}}$; otherwise, H is *non-scalar*;
- (ii) H is *accessible* if there is a sequence $\{H_n\}_{n\in\mathbb{N}}$ in $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ such that $H_n \neq H$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|H - H_n\| = 0$; otherwise, H is *isolated*;
- (iii) H is *essentially accessible* if there exist a unital normed algebra $\mathfrak{B}$, an isometric homomorphism $I\colon \mathfrak{A} \to \mathfrak{B}$ with $I(e_{\mathfrak{A}}) = e_{\mathfrak{B}}$, and a sequence $\{H_n\}_{n\in\mathbb{N}}$ in $\mathrm{Hom}(\mathfrak{X}, \mathfrak{B})$ such that $H_n \neq I \circ H$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|I \circ H - H_n\| = 0$; otherwise, H is *totally isolated*.

REMARK. The following observation complements the above definition: if $\mathfrak{X}$ is a normed algebra and if $\mathfrak{A}$ is a unital algebra, then, for each $\phi \in \Delta(\mathfrak{X})$, $e_{\mathfrak{A}} \otimes \phi$ is a non-zero, scalar homomorphism in $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$, and, conversely, every non-zero, scalar homomorphism in $\mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ takes the form $e_{\mathfrak{A}} \otimes \phi$ for some $\phi \in \Delta(\mathfrak{X})$.

THEOREM 2.14. *Let $\mathfrak{L}$ be an ordered AL-algebra with fundamental character l, and let $\mathfrak{A}$ be a unital normed algebra. Then the fundamental homomorphism $e_{\mathfrak{A}} \otimes l$ is totally isolated.*

Proof. Let $\mathfrak{B}$ be a unital normed algebra, and let $I\colon \mathfrak{A} \to \mathfrak{B}$ be an isometric homomorphism with $I(e_{\mathfrak{A}}) = e_{\mathfrak{B}}$. Then $I \circ (e_{\mathfrak{A}} \otimes l) = e_{\mathfrak{B}} \otimes l$. By Theorem 2.7, if $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{B})$ is such that $\|H - e_{\mathfrak{B}} \otimes l\| < 1$, then $H = e_{\mathfrak{B}} \otimes l$. The theorem follows. ∎

As it turns out, there exist totally isolated homomorphisms different from fundamental homomorphisms. With a view to revealing such homomorphisms, we next establish, in Theorem 2.16 below, a general result facilitating identification of totally isolated homomorphisms.

LEMMA 2.15. *Let $\mathfrak{A}$ be a unital normed algebra, let X be a normed space, and let $x^* \in X^*$ be such that $\|x^*\| = 1$. If e is an idempotent in $\mathfrak{A}$ such that $\|e \otimes x^* - e_{\mathfrak{A}} \otimes x^*\| < 1$, then $e = e_{\mathfrak{A}}$.*

Proof. Note that

$$\|e \otimes x^* - e_{\mathfrak{A}} \otimes x^*\| = \|e - e_{\mathfrak{A}}\| \, \|x^*\| = \|e - e_{\mathfrak{A}}\|.$$

Hence, by assumption, $\|e - e_{\mathfrak{A}}\| < 1$. As e is idempotent, we have $\|e^n - e_{\mathfrak{A}}\| = \|e - e_{\mathfrak{A}}\|$ for each $n \in \mathbb{N}$. Thus $\sup_{n\in\mathbb{N}} \|e^n - e_{\mathfrak{A}}\| < 1$, and now an application of Proposition 2.9 ensures that $e = e_{\mathfrak{A}}$. ∎

THEOREM 2.16. *Let $\mathfrak{X}$ be a normed algebra, and let ϕ be a character on $\mathfrak{X}$ with $\|\phi\| = 1$. Suppose, moreover, that $\mathfrak{X}$ has the following property:*

(P_ϕ) *if $\mathfrak{B}$ is a unital normed algebra and $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{B})$ is such that $\|H - e_{\mathfrak{B}} \otimes \phi\| < 1$, then there exists an idempotent e in $\mathfrak{B}$ such that $H = e \otimes \phi$.*

Then, for every unital normed algebra $\mathfrak{A}$, $e_{\mathfrak{A}} \otimes \phi$ is totally isolated.

We remark that when $\mathfrak{X}$ is unital, the assumption $\|\phi\| = 1$ is automatically satisfied. In general, however, only the inequality $\|\phi\| \leq 1$ holds—the value of $\|\phi\|$ can in fact turn out to be smaller than any given positive number (see, e.g., [15, p. 78]).

Proof of Theorem 2.16. Let $\mathfrak{A}$ be a unital normed algebra. Suppose that $\mathfrak{B}$ is a unital normed algebra and that $I\colon \mathfrak{A} \to \mathfrak{B}$ is an isometric homomorphism with $I(e_{\mathfrak{A}}) = e_{\mathfrak{B}}$. Then $I \circ (e_{\mathfrak{A}} \otimes \phi) = e_{\mathfrak{B}} \otimes \phi$, and, to complete the proof, it suffices to show that, if $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{B})$ is such that

$$\|H - e_{\mathfrak{B}} \otimes \phi\| < 1, \tag{2.2}$$

then $H = e_{\mathfrak{B}} \otimes \phi$. So, suppose that $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{B})$ obeys (2.2). By invoking property (P_ϕ), we deduce that there exists an idempotent e in $\mathfrak{B}$ such that $H = e \otimes \phi$. Now (2.2) can be restated as $\|e \otimes \phi - e_{\mathfrak{B}} \otimes \phi\| < 1$. Using the assumption that $\|\phi\| = 1$ and applying Lemma 2.15, we conclude that $e = e_{\mathfrak{B}}$. In other words, $H = e_{\mathfrak{B}} \otimes \phi$, as required. ∎

We next exhibit a wide class of essentially accessible homomorphisms and give, as a consequence, a characterisation of totally isolated homomorphisms. We start by recalling a definition.

Let $\mathfrak{A}$ be a unital normed algebra. For each $x \in \mathfrak{A}$, let L_x be the operator in $\mathscr{L}(\mathfrak{A})$ given by

$$L_x y = xy \qquad (y \in \mathfrak{A}).$$

It is readily verified that the mapping

$$L\colon \mathfrak{A} \to \mathscr{L}(\mathfrak{A}), \qquad x \mapsto L_x,$$

is a homomorphism from $\mathfrak{A}$ into $\mathscr{L}(\mathfrak{A})$. It is standard to term L the *left regular representation* of $\mathfrak{A}$ on $\mathfrak{A}$. The homomorphism L is isometric: it is clear that, for all $x \in \mathfrak{A}$, $\|L_x\| \leq \|x\|$, and, as $L_x e_{\mathfrak{A}} = x$, we in fact have $\|L_x\| = \|x\|$. Moreover, L is unital: the equality $L(e_{\mathfrak{A}}) = I_{\mathfrak{A}}$ holds. With these attributes, L is well suited to play the role of an isometric homomorphism required to establish essential accessibility of a homomorphism.

THEOREM 2.17. *Let $\mathfrak{X}$ be a normed algebra, and let $\mathfrak{A}$ be a unital normed algebra. If $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ is not scalar, then H is essentially accessible.*

Proof. Suppose that $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ is not scalar. We shall show that the composition $L \circ H$ can be approximated by members of $\mathrm{Hom}(\mathfrak{X}, \mathscr{L}(\mathfrak{A}))$ different from $L \circ H$.

By assumption, there exists $x_0 \in \mathfrak{X}$ such that $H(x_0)$ is not a scalar multiple of $e_{\mathfrak{A}}$. Consequently, $L_{H(x_0)}$ is not a scalar multiple of $I_{\mathfrak{A}}$. Applying a standard result from operator theory (see, e.g., [13, Lemma 1]), we deduce that $L_{H(x_0)}$ is not in the centre of $\mathscr{L}(\mathfrak{A})$, i.e., there exists B in $\mathscr{L}(\mathfrak{A})$ such that $L_{H(x_0)}B \neq BL_{H(x_0)}$. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers, each smaller than $\|B\|^{-1}$, such that $\epsilon_n \to 0$ as $n \to \infty$. Then, for each $n \in \mathbb{N}$, the operator

$$B_n := I_{\mathfrak{A}} - \epsilon_n B$$

is invertible, and we can define a homomorphism H_n in $\mathrm{Hom}(\mathfrak{X}, \mathscr{L}(\mathfrak{A}))$ by setting

$$H_n(x) = B_n^{-1} L_{H(x)} B_n \quad (x \in \mathfrak{X}).$$

Since $L_{H(x_0)}$ and B do not commute, it follows that $H_n(x_0) \neq L_{H(x_0)}$, and hence $H_n \neq L \circ H$ for all $n \in \mathbb{N}$.

If $x \in \mathfrak{X}$ and if $n \in \mathbb{N}$, then

$$\begin{aligned}\|L_{H(x)} - B_n^{-1} L_{H(x)} B_n\| &\leq \|L_{H(x)} - B_n^{-1} L_{H(x)}\| + \|B_n^{-1} L_{H(x)} - B_n^{-1} L_{H(x)} B_n\| \\ &\leq \|I_{\mathfrak{A}} - B_n^{-1}\| \, \|H(x)\| + \|B_n^{-1}\| \, \|H(x)\| \, \|I_{\mathfrak{A}} - B_n\|.\end{aligned}$$

Hence

$$\begin{aligned}\|L \circ H - H_n\| &= \sup_{\|x\|=1} \|L_{H(x)} - H_n(x)\| \\ &\leq \|I_{\mathfrak{A}} - B_n^{-1}\| \, \|H\| + \Big(\sup_{n\in\mathbb{N}} \|B_n^{-1}\|\Big) \|H\| \, \|I_{\mathfrak{A}} - B_n\|.\end{aligned}$$

Since $\lim_{n\to\infty} \|I_{\mathfrak{A}} - B_n\| = \lim_{n\to\infty} \|I_{\mathfrak{A}} - B_n^{-1}\| = 0$, which, in particular, implies that $\sup_{n\in\mathbb{N}} \|B_n^{-1}\| < \infty$, we conclude that $\lim_{n\to\infty} \|L \circ H - H_n\| = 0$. The theorem follows. ∎

Corollary 2.18. *Let $\mathfrak{X}$ be a normed algebra, and let $\mathfrak{A}$ be a unital normed algebra. If $H \in \mathrm{Hom}(\mathfrak{X}, \mathfrak{A})$ is totally isolated, then H is scalar.*

In light of this corollary, one might be tempted to think that every scalar homomorphism is totally isolated. But this is not the case. We shall soon reveal an ordered AL-algebra, some of whose scalar homomorphisms are totally isolated, and some are not (cf. Theorems 4.3 and 4.5). It is also to be noted that zero homomorphisms from ordered AL-algebras can be totally isolated in some cases, and accessible in others. Right below we identify circumstances under which the first eventuality occurs. Later on (see Section 4.1) we shall give an example of the appearance of the second possibility.

Let $\mathfrak{X}$ be a normed algebra. A net $\{u_\iota\}_{\iota\in I}$ in $\mathfrak{X}$ is called a *left* (respectively, *right, two-sided*) *approximate identity* for $\mathfrak{X}$ if

$$\lim_{\iota\in I} \|x - u_\iota x\| = 0 \quad \text{(respectively, } \lim_{\iota\in I} \|x - xu_\iota\| = 0, \ \lim_{\iota\in I}(\|x - u_\iota x\| + \|x - xu_\iota\|) = 0)$$

for every $x \in \mathfrak{X}$. It is said to be *bounded* if there is $K > 0$ such that $\|u_\iota\| \leq K$ for all $\iota \in I$; and, in this case, the least such K, $\sup_{\iota\in I} \|u_\iota\|$, is called the *bound* of $\{u_\iota\}_{\iota\in I}$. A (left, right, two-sided) approximate identity is *contractive* if it has bound 1. If $\mathfrak{X}$ is commutative, we speak just of a (*bounded*) *approximate identity.*

THEOREM 2.19. *Let $\mathfrak{X}$ be a normed algebra with a left (right) bounded approximate identity $\{u_\iota\}_{\iota\in I}$ with bound K, and let $\mathfrak{A}$ be a normed algebra. If $H \in \mathrm{Hom}(\mathfrak{X},\mathfrak{A})$ satisfies $\|H\| < K^{-1}$, then $H = 0$.*

Proof. We confine ourselves to the case where $\{u_\iota\}_{\iota\in I}$ is a left bounded approximate identity; the other case is handled similarly. Let $x \in \mathfrak{X}$. Then, for every $\iota \in I$ and every $n \in \mathbb{N}$,

$$x - u_\iota^n x = (u_\iota^{n-1} + \cdots + u_\iota + 1)(x - u_\iota x),$$

so that

$$\|x - u_\iota^n x\| \le C_{K,n}\|x - u_\iota x\|,$$

where $C_{K,n} := K^{n-1} + \cdots + K + 1$. Consequently,

$$\|H(x) - H(u_\iota^n x)\| \le \|H\|C_{K,n}\|x - u_\iota x\|.$$

Also

$$\|H(u_\iota^n x)\| = \|H(u_\iota)^n H(x)\| \le \|H(u_\iota)\|^n\|H(x)\| \le (\|H\|K)^n\|H(x)\|.$$

Thus

$$\|H(x)\| \le \|H\|C_{K,n}\|x - u_\iota x\| + (\|H\|K)^n\|H(x)\|.$$

Passing to the limit along the ordered set I, we obtain

$$\|H(x)\| \le (\|H\|K)^n\|H(x)\|.$$

Letting $n \to \infty$ and using $\|H\|K < 1$, we find that $H(x) = 0$. Now the theorem follows on account of the arbitrariness of x. ∎

COROLLARY 2.20. *Let $\mathfrak{X}$ be a normed algebra with a left (right) bounded approximate identity, and let $\mathfrak{A}$ be a normed algebra. Then the zero homomorphism from $\mathfrak{X}$ to $\mathfrak{A}$ is totally isolated.*

Save one, all the AL-algebras considered henceforth will have a two-sided contractive approximate identity; and, accordingly, all zero homomorphisms from these algebras will be totally isolated.

3. Semigroup algebras

A distinctive class of ordered AL-algebras is constituted by the semigroup algebras of subsemigroups of locally compact groups, where each subsemigroup is Haar measurable, not locally null, and such that the neutral element of the respective ambient group is a density point of the subsemigroup. In this chapter, we shall be concerned with isolability properties of homomorphisms from algebras in this class. The reason for singling out the said class amongst all semigroup algebras is that then every member algebra is ensured to possess a contractive two-sided approximate identity.

3.1. Background. The material of this section is divided into three subsections.

3.1.1. Characters and semi-characters. Let S be a semigroup, with product denoted by juxtaposition. A *semi-character* on S (as defined, say, in [39]) is a bounded, complex-valued function ζ on S, not identically zero, which satisfies $\zeta(st) = \zeta(s)\zeta(t)$ for all $s, t \in S$. If ζ is a semi-character on S, then, necessarily, $|\zeta(s)| \leq 1$ for all $s \in S$ (cf. [83, Lemma A.27]). A complex-valued function f on a set A satisfying $|f(a)| = 1$ for all $a \in A$ is said to be *unitary*. A *character* on S is a unitary semi-character on S. The constant function 1_S is the *trivial* character on S.

Let S be a topological semigroup. We denote by $\widehat{S}$ the set of all continuous semi-characters on S, and by $\widehat{S}_{\mathrm{u}}$ the set of all continuous characters on S. The set of all non-negative, real-valued, continuous semi-characters on S will be denoted by $\widehat{S}_+$. Clearly, $\widehat{S}_+ = \{|\zeta| \mid \zeta \in \widehat{S}\}$.

Let G be a locally compact Abelian group, and let $\widehat{G}$ be its Pontryagin dual, that is, the set of all continuous characters on G, endowed with the compact-open topology; it forms an Abelian group under the pointwise multiplication of characters. Let S be a subsemigroup of G. Endowed with the subspace topology, S is a topological semigroup. It is clear that the restriction of any character in $\widehat{G}$ to S is a character in $\widehat{S}_{\mathrm{u}}$. A remarkable converse fact is that:

(i) every character in $\widehat{S}_{\mathrm{u}}$ is the restriction to S of some character in $\widehat{G}$;
(ii) every semi-character in $\widehat{S}$ can be represented (generally non-uniquely) as a product $\rho\sigma$, where $\rho \in \widehat{S}_+$ and $\sigma \in \widehat{S}_{\mathrm{u}}$;

see [24, Proposition 4.4]. When S is a closed subgroup of G, and as such is locally compact in the subspace topology, we have $\widehat{S} = \widehat{S}_{\mathrm{u}}$; in other words, $\widehat{S}$ is identical with the dual group of S (cf. [83, Lemma 3.4]). Note that confining oneself to the consideration of closed subgroups only is not a restriction here: every locally compact subgroup of a Hausdorff group is necessarily closed; see [38, Theorem 5.11] or [69, Proposition 1-6].

3.1.2. Semigroup algebras of subsemigroups of locally compact groups. Let G be a locally compact group. Let m_G denote a fixed, but arbitrary, left Haar measure on G. The *group algebra* of G, $L^1(G)$, is the Banach algebra of equivalence classes of complex-valued, Haar integrable functions on G, with the norm

$$\|f\|_1 = \int_G |f(s)| \,\mathrm{d}m_G(s) \qquad (f \in L^1(G))$$

and with the convolution as multiplication defined by

$$(f \star g)(s) = \int_G f(t)g(t^{-1}s) \,\mathrm{d}m_G(t) \qquad (f, g \in L^1(G), \text{ a.e. } s \in G).$$

If, for real-valued members f and g of $L^1(G)$, we define $f \leq g$ to mean that $f(s) \leq g(s)$ for almost every $s \in G$, then $L^1(G)$ becomes a Banach lattice. It is elementary to check that $L^1(G)$ is in fact a complex ordered AL-algebra.

We recall that a Haar measurable subset E of G is *locally null* if $m_G(E \cap K) = 0$ for each compact subset K of G. This definition does not change if m_G in it is replaced by any positive multiple of m_G, or by a right Haar measure on G. A property of points of

G is said to hold *locally almost everywhere* if the set of points at which it fails to hold is locally null. If a Haar measurable subset of G is locally null, then its left (right) Haar measure is equal either to 0 or to ∞. In the case where G is σ-compact (that is, in the case where G is a countable union of compact sets), every locally null subset of G has null left (right) Haar measure. Accordingly, when G is σ-compact, to say that a Haar measurable subset E of G is not locally null amounts to saying that E has positive left (right) Haar measure.

Let S be a non-locally-null, Haar measurable subsemigroup of G. The Banach space $L^1(S)$ of equivalence classes of complex-valued, Haar integrable functions on S, with the norm

$$\|f\|_1 = \int_S |f(s)| \, \mathrm{d}m_G(s) \quad (f \in L^1(S)),$$

is a Banach algebra for the convolution product defined by

$$\int_S h(f \star g) \, \mathrm{d}m_G = \int_{S\times S} h(st)f(s)g(t) \, \mathrm{d}(m_G \otimes m_G)(s,t) \qquad (f, g \in L^1(S),\ h \in L^\infty(S)).$$

Here $L^\infty(S)$ denotes the Banach space of equivalence classes of essentially bounded, Haar measurable, complex-valued functions (with functions identified if they are equal locally almost everywhere) on S. The assumption that S is not locally null ensures that $L^1(S)$ does not reduce to the class of the zero function. An equivalent definition of the convolution product can be formulated as follows. Given $f \in L^1(S)$, let $\tilde{f}$ be the element of $L^1(G)$ equal almost everywhere to f on S and almost everywhere to zero outside S. Then, for $f, g \in L^1(S)$, the convolution $f \star g$ can be defined as the restriction of $\tilde{f} \star \tilde{g}$ to S. Even more explicitly,

$$(f \star g)(s) = \int_{S \cap sS^{-1}} f(t)g(t^{-1}s) \, \mathrm{d}m_G(t) \quad (\text{a.e. } s \in S), \tag{3.1}$$

where $sS^{-1} = \{st^{-1} \mid t \in S\}$. The algebra $L^1(S)$ is termed the *semigroup algebra* of S. It is readily seen that $L^1(S)$ is a complex ordered AL-algebra.

It is worth mentioning that one can also define $L^1(S)$ as a *vanishing algebra*, that is, a subset of $L^1(G)$ which consists of all functions vanishing almost everywhere on the complement of a Haar-measurable subset of G, and which is closed under convolution. One can namely put

$$L^1(S) = \{f \in L^1(G) \mid f(s) = 0 \text{ for a.e. } s \in G \setminus S\};$$

see, e.g., [24], [53], [55], [56], [57], [70], [79], [80], [81]. Viewed as a vanishing algebra, $L^1(S)$ is a closed subalgebra of $L^1(G)$. Here we shall not follow this alternative path.

3.1.3. Semigroup algebras of discrete semigroups. Given a set Γ, let $\ell^1(\Gamma)$ be the Banach space of all complex-valued functions f on Γ such that $\sum_{\gamma\in\Gamma} |f(\gamma)| < \infty$, furnished with the norm

$$\|f\|_1 = \sum_{\gamma\in\Gamma} |f(\gamma)| \quad (f \in \ell^1(\Gamma)).$$

For each $\gamma \in \Gamma$, we denote by δ_γ the characteristic function of $\{\gamma\}$ when the latter is viewed as a member of $\ell^1(\Gamma)$. One can think of any element f of $\ell^1(\Gamma)$ as the generalised sum $\sum_{\gamma\in\Gamma} f(\gamma)\delta_\gamma$, where $\sum_{\gamma\in\Gamma} |f(\gamma)| < \infty$.

When S is a (discrete) semigroup, $\ell^1(S)$ is a Banach algebra for the convolution multiplication defined by

$$\delta_s \star \delta_t = \delta_{st} \quad (s, t \in S).$$

Thus

$$(f \star g)(s) = \sum_{\substack{pr=s \\ p,r\in S}} f(p)g(r) \quad (f, g \in \ell^1(S),\ s \in S), \tag{3.2}$$

where the sum is taken to be 0 when there are no $p, r \in S$ with $pr = s$. The algebra $\ell^1(S)$ is called the *semigroup algebra* of S.

When G is a discrete group and the (two-sided) Haar measure on G is counting measure, $L^1(G)$ is the same as $\ell^1(G)$. Moreover, if S is a subsemigroup of G, then S is obviously not locally null, and the convolution product in $\ell^1(S)$ defined in (3.2) is the same as the convolution product given in (3.1).

3.2. Semigroup algebras with α-number 1. We now exhibit a class of semigroup algebras for which the α-number is equal to 1. Three specific algebras in this class will be the subject of discussion in the next chapter.

Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Given $f \in L^1(S)$, the *generalised Laplace transform* of f is the function $\zeta \mapsto \mathscr{L}_\zeta(f)$ on $\widehat{S}$ defined by

$$\mathscr{L}_\zeta(f) = \int_S f(s)\zeta(s)\,\mathrm{d}m_G(s) \quad (\zeta \in \widehat{S}).$$

It is standard to verify that, for each $\zeta \in \widehat{S}$, the mapping $\mathscr{L}_\zeta\colon f \mapsto \mathscr{L}_\zeta(f)$ is a character on $L^1(S)$. Conversely, every character on $L^1(S)$ coincides with $\mathscr{L}_\zeta$ for some uniquely determined $\zeta \in \widehat{S}$ (cf. [24, Theorem 5.1] or [71, Theorem 13]). It is immediate that $\mathscr{L}_{1_S}$ is the fundamental character on $L^1(S)$.

Lemma 3.1. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. If $\rho \in \widehat{S}_+ \setminus \{1_S\}$ and if $\sigma \in \widehat{S}_{\mathrm{u}}$, then $\rho\sigma$ is in $\widehat{S}$ and $\|\mathscr{L}_\sigma - \mathscr{L}_{\rho\sigma}\| = 1$.*

Proof. It is clear that $\rho\sigma$ is a homomorphism from S into $\overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is considered as a semigroup under multiplication. As σ is unitary and ρ is not identically zero, $\rho\sigma$ is not identically zero. Thus $\rho\sigma$ is in $\widehat{S}$.

If $f \in L^1(S)$, then

$$\begin{aligned} |\mathscr{L}_\sigma(f) - \mathscr{L}_{\rho\sigma}(f)| &\le \int_S |f(s)(1-\rho(s))\sigma(s)|\,\mathrm{d}m_G(s) \\ &= \int_S |f(s)|(1-\rho(s))\,\mathrm{d}m_G(s) \\ &\le \int_S |f(s)|\,\mathrm{d}m_G(s) = \|f\|_1, \end{aligned}$$

where the equality in the second line follows from σ being unitary and the fact that $0 \le \rho(s) \le 1$ for all $s \in S$. This implies that $\|\mathscr{L}_\sigma - \mathscr{L}_{\rho\sigma}\| \le 1$.

To prove the reverse inequality, we first claim that the set

$$E := \{s \in S \mid \rho(s) < 1\}$$

is not locally null. For assume that this is false. Then ρ is equal to 1 locally almost everywhere. Select, arbitrarily, a compact subset C of S with positive (and necessarily finite) Haar measure. If $s \in S$, then

$$\rho(s) \int_C \rho(t) \, \mathrm{d}m_G(t) = \int_C \rho(st) \, \mathrm{d}m_G(t) = \int_{sC} \rho(t) \, \mathrm{d}m_G(t),$$

where $sC = \{st \mid t \in C\}$. Since, in particular, ρ is equal to 1 almost everywhere on C and on sC, the equality between the leftmost and rightmost expressions above can be rewritten as

$$\rho(s) m_G(C) = m_G(sC).$$

Now, since $m_G(sC) = m_G(C)$, we have

$$\rho(s) m_G(C) = m_G(C),$$

and hence $\rho(s) = 1$. But this contradicts the hypothesis that $\rho \ne 1_S$. Thus the claim holds.

For each $0 < r < 1$, let

$$E_r := \{s \in S \mid \rho(s) \le r\}.$$

Since

$$E = \bigcup_{r \in (0,1) \cap \mathbb{Q}} E_r$$

and since E is not locally null, there exists $r \in (0,1) \cap \mathbb{Q}$ such that E_r is not locally null. Let K be a compact subset of S such that $m_G(E_r \cap K) > 0$. Since $m_G(E_r \cap K) \le m_G(K) < \infty$ and since Haar measure is inner regular on Haar measurable sets of finite measure (cf. [38, Theorem 11.31]), there is a compact subset L of $E_r \cap K$ with positive (finite) Haar measure. Given $n \in \mathbb{N}$, let

$$L^{(n)} = \{s_1 \dots s_n \mid s_1, \dots, s_n \in L\}.$$

Then $L^{(n)}$ is a compact subset of S. By Steinhaus' theorem [82] on the Minkowski product of two sets, $L^{(n)}$ contains an open subset, and hence has positive Haar measure. We recall that the theorem in question, stated in a form relevant to the present context, asserts that, if A and B are Haar measurable subsets of a locally compact group, each of positive finite Haar measure, then $AB = \{ab \mid a \in A,\, b \in B\}$ has non-empty interior (see [9] and [38, Theorem 20.17]). Set

$$f_n = \frac{1}{m_G(L^{(n)})} \chi_{L^{(n)}} \quad \text{and} \quad g_n = \overline{\sigma} f_n \quad (n \in \mathbb{N}).$$

For each $n \in \mathbb{N}$, we have $\|f_n\|_1 = \|g_n\|_1 = 1$, and, since $\rho(s) \le r^n$ whenever $s \in L^{(n)}$, we also have

$$\mathscr{L}_\sigma(g_n) - \mathscr{L}_{\sigma\rho}(g_n) = \int_S f_n(s)(1-\rho(s))\,\mathrm{d}m_G(s)$$
$$= \frac{1}{m_G(L^{(n)})}\int_{L^{(n)}}(1-\rho(s))\,\mathrm{d}m_G(s) \geq 1-r^n.$$

Hence $\|\mathscr{L}_\sigma - \mathscr{L}_{\rho\sigma}\| \geq 1-r^n$. Letting $n\to\infty$, we obtain $\|\mathscr{L}_\sigma - \mathscr{L}_{\rho\sigma}\| \geq 1$, as desired. ■

Theorem 3.2. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G. Suppose that $\widehat{S}_+\setminus\{1_S\}$ is non-empty. Then $\alpha(L^1(S)) = 1$.*

Proof. Let $\mathfrak{A}$ be a unital normed algebra. Pick $\rho\in\widehat{S}_+\setminus\{1_S\}$. Then $e_\mathfrak{A}\otimes\mathscr{L}_\rho \neq e_\mathfrak{A}\otimes\mathscr{L}_{1_S}$. Moreover, an application of Lemma 3.1 with $\sigma = 1_S$ yields

$$\|e_\mathfrak{A}\otimes\mathscr{L}_{1_S} - e_\mathfrak{A}\otimes\mathscr{L}_\rho\| = \|\mathscr{L}_{1_S} - \mathscr{L}_\rho\| = 1.$$

Hence $\alpha(L^1(S)) \leq 1$. On the other hand, Theorem 2.7 guarantees that $\alpha(L^1(S)) \geq 1$. Combining these two inequalities completes the proof. ■

3.3. A representation result. We now present a technical result concerning the form of homomorphisms from semigroup algebras of a certain type. The relevant semigroups algebras here will be exactly the semigroup algebras described in the opening paragraph of this chapter.

Let G be a locally compact group. An element s in G is a *density point* of a Haar measurable subset A of G if every open neighbourhood of s meets A in a set of positive Haar measure. The set of all density points of A will be denoted by $D(A)$. Simon [81] has shown that, if $A\subset G$ is Haar measurable, then:

(i) $D(A)$ is closed;
(ii) $A^\circ \subset D(A) \subset \overline{A}$;
(iii) $A\setminus D(A)$ is locally null.

In the above statement, A° denotes the interior of A, and $\overline{A}$ denotes the closure of A. The same author has further shown that, if $S\subset G$ is a Haar measurable semigroup, then $D(S)$ is also a semigroup. Our interest in density points arises in connection with the question of the existence of approximate identities for semigroup algebras.

Throughout, for a locally compact group G written multiplicatively (which is done by default), the neutral element of G will be denoted by e_G.

Lemma 3.3. *Let G be a locally compact group, and let S be a non-locally-null, Haar measurable subsemigroup of G such that $e_G\in D(S)$. Then $L^1(S)$ has a two-sided approximate identity $\{u_\iota\}_{\iota\in I}$ such that $\|u_\iota\| = 1$ for each $\iota\in I$.*

Proof. Let $\{V_\iota\}_{\iota\in I}$ be a base of precompact (closure is compact) neighbourhoods of e_G in G. This can be viewed as a net directed downward by inclusion. For each $\iota\in I$, let

$$u_\iota = \frac{1}{m_G(V_\iota\cap S)}\chi_{V_\iota\cap S}.$$

The assumption that $e_G\in D(S)$ guarantees that the denominator of each fraction is positive. It is clear that $\|u_\iota\| = 1$ for each $\iota\in I$. Also, it is routine to verify that $\{u_\iota\}_{\iota\in I}$ is a two-sided approximate identity for $L^1(S)$ (cf. [66, pp. 528–529]). Incidentally, checking that $\{u_\iota\}_{\iota\in I}$ is a right approximate identity is somewhat more involved than checking that $\{u_\iota\}_{\iota\in I}$ is a left approximate identity, as it requires the use of the modular function relating right Haar measure to left Haar measure. ∎

Let A be a topological space. A family $\{F(a)\}_{a\in A}$ of bounded linear operators on a Banach space X is *strongly continuous* if, for each $x \in X$, the function $A \ni a \mapsto F(a)x \in X$ is continuous in norm.

Let S be a semigroup. In the case where S is unital, the neutral element of S is denoted by e_S. Let $\mathfrak{A}$ be a unital algebra. An $\mathfrak{A}$-valued family $\{\mathscr{S}(s)\}_{s\in S}$ is a *semigroup* in $\mathfrak{A}$ if:

(i) $\mathscr{S}(s)\mathscr{S}(t) = \mathscr{S}(st)$ for all $s, t \in S$;
(ii) $\mathscr{S}(e_S) = e_{\mathfrak{A}}$ whenever S is unital.

An $\mathscr{L}(X)$-valued semigroup, where X is a non-zero normed space, is called a semigroup on X.

Let G be a locally compact group. Suppose that S is a non-locally-null, Haar measurable subsemigroup of G. Given $s \in S$, we denote by S_s the operator of *right shift* by s on $L^1(S)$, defined by

$$(S_s f)(t) = \begin{cases} f(s^{-1}t) & \text{for a.e. } t \in sS, \\ 0 & \text{for a.e. } t \in S \setminus sS \end{cases} \qquad (f \in L^1(S)).$$

It is straightforward to verify that the family $\{S_s\}_{s\in S}$ is a strongly continuous semigroup on $L^1(S)$ and, moreover, that

$$S_s(f \star g) = S_s f \star g \tag{3.3}$$

holds for all $s \in S$ and all $f, g \in L^1(S)$.

Given a mapping $f\colon A \to B$, where A and B are sets, we denote by $\operatorname{Ran}(f)$ the range of f.

Let X be a normed space. Given $S \in \mathscr{L}(X)$ and a linear subspace $Y \subset X$ such that $S(Y) \subset Y$, we denote by $S{\restriction_Y}$ the restriction of S to Y, that is, the linear operator in $\mathscr{L}(Y)$ defined by

$$(S{\restriction_Y})y = Sy \qquad (y \in Y).$$

Given a mapping $f\colon A \to \mathscr{L}(X)$, where A is a set, and a linear subspace $Y \subset X$ such that $(f(a))(Y) \subset Y$ for each $a \in A$, we denote by $f{\restriction_Y}$ the mapping

$$f{\restriction_Y}\colon A \to \mathscr{L}(Y), \qquad a \mapsto f(a){\restriction_Y}.$$

It will be now convenient to make two remarks on what is essentially a matter of disentangling and clarifying notation, one aiming at immediate, and the other at postponed use. Let $\mathfrak{X}$ be a normed algebra, and let $\mathfrak{A}$ be a unital normed algebra. Let L continue to denote the left regular representation of $\mathfrak{A}$ on $\mathfrak{A}$. First, suppose that H is a homomorphism in $\operatorname{Hom}(\mathfrak{X}, \mathfrak{A})$. Then

$$\big((L \circ H)(x)\big)H(y) = L_{H(x)}H(y) = H(x)H(y) = H(xy)$$

for all $x, y \in \mathfrak{X}$. This shows that

$$\big((L \circ H)(x)\big)(\operatorname{Ran}(H)) \subset \operatorname{Ran}(H)$$

for each $x \in \mathfrak{X}$, and we may therefore consider the mapping

$$(L \circ H)\restriction_{\mathrm{Ran}(H)} \colon \mathfrak{X} \to \mathscr{L}(\mathrm{Ran}(H)).$$

It is immediate that

$$\big((L \circ H)\restriction_{\mathrm{Ran}(H)}\big)(x)z = H(x)z$$

for each $x \in \mathfrak{X}$ and each $z \in \mathrm{Ran}(H)$.

Second, suppose that ϕ is a functional in $\Delta(\mathfrak{X})$ and that Y is a linear subspace of $\mathfrak{A}$. Then

$$\big(L \circ (e_{\mathfrak{A}} \otimes \phi)\big)(x)y = L_{(e_{\mathfrak{A}} \otimes \phi)(x)}y = (e_{\mathfrak{A}} \otimes \phi)(x)y = \phi(x)y$$

for each $x \in \mathfrak{X}$ and each $y \in Y$. This implies that

$$\big(\big(L \circ (e_{\mathfrak{A}} \otimes \phi)\big)(x)\big)(Y) \subset Y$$

for each $x \in \mathfrak{X}$, and we may meaningfully consider the mapping

$$(L \circ (e_{\mathfrak{A}} \otimes \phi))\restriction_{Y} \colon \mathfrak{X} \to \mathscr{L}(Y).$$

It is clear that

$$\big((L \circ (e_{\mathfrak{A}} \otimes \phi))\restriction_{Y}\big)(x)y = \phi(x)y$$

for each $x \in \mathfrak{X}$ and each $y \in Y$.

We are ready to present the main result of this section. This result is inspired by the work of Kisyński on the so-called algebraic version of the Hille–Yosida theorem (cf. [48, Theorems 4.2 and 5.5], [49, Theorems 10.2 and 12.5], and also [16, Theorems 3.3 and 4.1]).

Proposition 3.4. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G such that $e_G \in D(S)$, and let $\mathfrak{A}$ be a unital normed algebra. Let $H\colon L^1(S) \to \mathfrak{A}$ be a non-zero, continuous homomorphism. Then:*

(i) *there exists a unique semigroup $\mathscr{S} = \{\mathscr{S}(s)\}_{s\in S}$ on $\mathrm{Ran}(H)$ such that*
$$\mathscr{S}(s)H(f) = H(S_s f) \tag{3.4}$$
for each $s \in S$ and each $f \in L^1(S)$; the semigroup $\mathscr{S}$ is strongly continuous and satisfies $\sup_{s\in S}\|\mathscr{S}(s)\| \le \|H\|$;

(ii) *if $\{u_\iota\}_{\iota\in I}$ is a left approximate identity for $L^1(S)$, then*
$$\mathscr{S}(s)x = \lim_{\iota\in I} H(S_s u_\iota)x \tag{3.5}$$
for each $s \in S$ and each $x \in \mathrm{Ran}(H)$;

(iii) *with L denoting the left regular representation of $\mathfrak{A}$, the mapping $(L \circ H)\restriction_{\mathrm{Ran}(H)}$ admits the representation*
$$\big((L \circ H)\restriction_{\mathrm{Ran}(H)}\big)(f)x =: H(f)x = \int_S f(s)\mathscr{S}(s)x \,\mathrm{d}m_G(s) \tag{3.6}$$
for each $f \in L^1(S)$ and each $x \in \mathrm{Ran}(H)$.

Proof. (i) First note that since H is non-zero, so too is $\mathrm{Ran}(H)$. Next, given $s \in S$, define a linear operator $\mathscr{S}(s)$ on $\mathrm{Ran}(H)$ by

$$\mathscr{S}(s)H(f) = H(S_s f) \quad (f \in L^1(S)).$$

To check that the above definition is correct, we need to show that, if $f, g \in L^1(S)$ are such that $H(f) = H(g)$, then

$$H(S_s f) = H(S_s g). \tag{3.7}$$

Choose a left approximate identity $\{u_\iota\}_{\iota \in I}$ for $L^1(S)$; that this can be done is ensured by Lemma 3.3. Note that

$$H(S_s u_\iota)H(f) = H(S_s u_\iota)H(g)$$

for all $\iota \in I$. Hence

$$\lim_{\iota \in I} H(S_s u_\iota)H(f) = \lim_{\iota \in I} H(S_s u_\iota)H(g) \tag{3.8}$$

provided that the limits exist. In view of (3.3),

$$H(S_s u_\iota)H(f) = H(S_s u_\iota \star f) = H(S_s(u_\iota \star f))$$

for each $\iota \in I$. Therefore

$$\begin{aligned}\lim_{\iota \in I} H(S_s u_\iota)H(f) &= \lim_{\iota \in I} H(S_s(u_\iota \star f)) = H(\lim_{\iota \in I} S_s(u_\iota \star f)) \\ &= H\Big(S_s\Big(\lim_{\iota \in I}(u_\iota \star f)\Big)\Big) = H(S_s f). \end{aligned} \tag{3.9}$$

Likewise

$$\lim_{\iota \in I} H(S_s u_\iota)H(g) = H(S_s g). \tag{3.10}$$

Putting (3.8)–(3.10) together yields (3.7), as required.

To proceed further, we refine our choice of the left approximate identity for $L^1(S)$ and now suppose that $\{u_\iota\}_{\iota \in I}$ satisfies $\|u_\iota\|_1 \leq 1$ for each $\iota \in I$; here again Lemma 3.3 ensures the feasibility of such a refinement. If $s \in S$ and if $f \in L^1(S)$, then

$$\|H(S_s u_\iota)H(f)\| \leq \|H(S_s u_\iota)\| \, \|H(f)\| \leq \|H\| \, \|S_s u_\iota\|_1 \|H(f)\| \leq \|H\| \, \|H(f)\|$$

for each $\iota \in I$. Hence, by (3.9),

$$\|H(S_s f)\| \leq \|H\| \, \|H(f)\|,$$

and this implies that $\mathscr{S}(s)$ is bounded, with norm no greater than $\|H\|$. Consequently, $\sup_{s \in S} \|\mathscr{S}(s)\| \leq \|H\|$.

Using the semigroup property of $\{S_s\}_{s \in S}$, one verifies at once that $\{\mathscr{S}(s)\}_{s \in S}$ is a semigroup on $\mathrm{Ran}(H)$. The strong continuity of $\mathscr{S}$ follows immediately from the strong continuity of $\{S_s\}_{s \in S}$, and the uniqueness of $\mathscr{S}$ follows immediately from the defining property of (3.4).

(ii) Formula (3.5) is a restatement of (3.9).

(iii) Let $f, g \in L^1(S)$. The function $S \ni s \mapsto S_s g \in L^1(S)$ is continuous, and the function $S \ni s \mapsto f(s)S_s g \in L^1(S)$ is dominated in norm by the non-negative Haar integrable function $s \mapsto |f(s)| \, \|g\|_1$. Therefore the function $S \ni s \mapsto f(s)S_s g \in L^1(S)$ is Bochner integrable. It is readily verified that

$$\int_S f(s) S_s g \, \mathrm{d}m_G(s) = f \star g,$$

where the integral is understood as a Bochner integral. Let $\mathfrak{B}$ be the Banach algebra completion of $\mathrm{Ran}(H)$. Using the interchangeability of the Bochner integral with bounded linear operators (cf. [97, p. 134, Corollary 2]), we have

$$\begin{aligned}\int_S f(s)\mathscr{S}(s)H(g)\,\mathrm{d}m_G(s) &= \int_S f(s)H(S_s g)\,\mathrm{d}m_G(s) = H\Big(\int_S f(s)S_s g\,\mathrm{d}m_G(s)\Big)\\ &= H(f\star g) = H(f)H(g),\end{aligned}$$

where the leftmost term is a priori a Bochner integral with values in $\mathfrak{B}$, but in fact is a member of $\mathrm{Ran}(H)$, as the first equality in the second line attests. This establishes (3.6). ∎

3.4. Applications. We now use the material from the last and previous sections to discuss isolability properties of homomorphisms.

The following is our key result.

THEOREM 3.5. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G such that $e_G \in D(S)$, and let $\mathfrak{A}$ be a unital normed algebra. If $\sigma \in \widehat{S}_{\mathrm{u}}$ and if $H\colon L^1(S) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$, then $H = e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$.*

Proof. The proof is divided into two steps. The first step shows that the homomorphisms $(L\circ H)\restriction_{\mathrm{Ran}(H)}$ and $(L\circ(e_{\mathfrak{A}}\otimes\mathscr{L}_\sigma))\restriction_{\mathrm{Ran}(H)}$, where L denotes the left regular representation of $\mathfrak{A}$, coincide. The second step upgrades the result of the first step to the equality of H and $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$.

STEP 1. Since $\|\mathscr{L}_\sigma\| = \|\sigma\|_\infty = 1$, where $\|\cdot\|_\infty$ denotes the uniform norm (on G), we have $\|e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| = \|e_{\mathfrak{A}}\|\,\|\mathscr{L}_\sigma\| = 1$; and since $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$ by assumption, we see that H is non-zero. Select a contractive two-sided approximate identity $\{u_\iota\}_{\iota\in I}$ for $L^1(S)$; this—we recall—can done by appealing to Lemma 3.3. Let $\{\mathscr{S}(s)\}_{s\in S}$ be the semigroup on $\mathrm{Ran}(H)$ whose existence is asserted in Proposition 3.4. If $s \in S$, if $x \in \mathrm{Ran}(H)$, and if $\iota \in I$, then

$$\begin{aligned}\|H(S_s u_\iota)x - (e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma)(S_s u_\iota)x\| &\le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|\,\|S_s u_\iota\|_1\|x\|\\ &\le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|\,\|x\|.\end{aligned}\tag{3.11}$$

Since σ is continuous, we have $\lim_{\iota\in I}\mathscr{L}_\sigma(S_s u_\iota) = \sigma(s)$, and so

$$\lim_{\iota\in I}(e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma)(S_s u_\iota)x = \sigma(s)x.$$

Combining this with (3.5), we deduce from (3.11) that

$$\|\mathscr{S}(s)x - \sigma(s)x\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|\,\|x\|.$$

Bearing in mind that σ is unitary, we further conclude that

$$\|\overline{\sigma(s)}\mathscr{S}(s)x - x\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|\,\|x\|.$$

Hence

$$\sup_{s\in S}\|\overline{\sigma(s)}\mathscr{S}(s) - I_{\mathrm{Ran}(H)}\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\|.$$

This together with the assumption that $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$ yields

$$\sup_{s\in S}\|\overline{\sigma(s)}\mathscr{S}(s) - I_{\mathfrak{B}}\| < 1.$$

An application of Proposition 2.9 now implies that $\overline{\sigma(s)}\mathscr{S}(s) = I_{\mathrm{Ran}(H)}$ for all $s \in S$. Hence, immediately, $\mathscr{S}(s) = \sigma(s)I_{\mathrm{Ran}(H)}$ for all $s \in S$, and further, by (3.6),

$$H(f)x = \mathscr{L}_\sigma(f)x \tag{3.12}$$

for all $f \in L^1(S)$ and all $x \in \mathrm{Ran}(H)$. Note that the latter equality can be phrased as the equality of $(L \circ H)\restriction_{\mathrm{Ran}(H)}$ and $(L \circ (e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma))\restriction_{\mathrm{Ran}(H)}$.

STEP 2. If $f \in L^1(S)$, then

$$\lim_{\iota\in I} H(f)H(u_\iota) = \lim_{\iota\in I} H(f * u_\iota) = H(f).$$

For each $\iota \in I$, putting $x = H(u_\iota)$ in (3.12) yields

$$H(f)H(u_\iota) = \mathscr{L}_\sigma(f)H(u_\iota).$$

Therefore

$$H(f) = \lim_{\iota\in I} \mathscr{L}_\sigma(f)H(u_\iota). \tag{3.13}$$

Choose $g \in L^1(S)$ so that $\mathscr{L}_\sigma(g) \neq 0$; the existence of such g follows, for example, from the observation that $\lim_{\iota\in I} \mathscr{L}_\sigma(S_s u_\iota) = \sigma(s) \neq 0$ for any $s \in S$. Taking f in (3.13) to be g, we find that the limit

$$e := \lim_{\iota\in I} H(u_\iota)$$

exists and is given by

$$e = (\mathscr{L}_\sigma(g))^{-1}H(g).$$

This last representation shows, notably, that e is a member of $\mathrm{Ran}(H)$. Note that e will not change if g is replaced by any other function $h \in L^1(S)$ satisfying $\mathscr{L}_\sigma(h) \neq 0$. In particular, given that $\mathscr{L}_\sigma(g * g) = (\mathscr{L}_\sigma(g))^2 \neq 0$, we may use $g * g$ in place of g. But

$$(\mathscr{L}_\sigma(g * g))^{-1}H(g * g) = ((\mathscr{L}_\sigma(g))^2)^{-1}(H(g))^2 = \big((\mathscr{L}_\sigma(g))^{-1}H(g)\big)^2,$$

and this means that e is an idempotent. Since (3.13) can be written as

$$H(f) = \mathscr{L}_\sigma(f)e = (e \otimes \mathscr{L}_\sigma)(f)$$

for each $f \in L^1(S)$, we have $H = e \otimes \mathscr{L}_\sigma$. Taking into account that the assumption about H can now be reformulated as $\|e \otimes \mathscr{L}_\sigma - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$, and that $\|\mathscr{L}_\sigma\| = 1$, we deduce from Lemma 2.15 that $e = e_{\mathfrak{A}}$. Hence, finally, $H = e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$. ∎

REMARK 3.6. In general, the inequality $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| < 1$ in Theorem 3.5 is optimal and cannot be replaced by the weaker inequality $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| \leq 1$. This stems from the fact that there always exists $H \in \mathrm{Hom}(L^1(S), \mathfrak{A})$ obeying $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| = 1$, and hence also differing from $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma$. One such H is the zero homomorphism from $L^1(S)$ into $\mathfrak{A}$ (cf. Theorem 2.7). Remarkably, H may be non-zero and satisfy $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| = 1$. This follows from the following observation: if S is a subsemigroup of G such that $\widehat{S}_+ \setminus \{1_S\}$ is non-empty, then, for every $\rho \in \widehat{S}_+ \setminus \{1_S\}$ and every $\sigma \in \widehat{S}_{\mathrm{u}}$, we have $\mathscr{L}_{\rho\sigma} \neq \mathscr{L}_\sigma$, and, by Lemma 3.1,

$$\|e_{\mathfrak{A}} \otimes \mathscr{L}_{\rho\sigma} - e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma\| = \|\mathscr{L}_{\rho\sigma} - \mathscr{L}_\sigma\| = 1.$$

Specific examples of semigroups S satisfying the assumptions of Theorem 3.5 and such that $\widehat{S}_+ \setminus \{1_S\} \neq \emptyset$ include $\mathbb{Z}^+$, $\mathbb{R}^+_{\mathrm{d}}$, and $\mathbb{R}^+$. See the next chapter for details.

Our next theorem reveals totally isolated homomorphisms that are different from fundamental homomorphisms.

Theorem 3.7. *Let G be a locally compact group, let S be a non-locally-null, Haar measurable subsemigroup of G such that $e_G \in D(S)$, and let $\mathfrak{A}$ be a unital normed algebra. If $\sigma \in \widehat{S}_{\mathrm{u}}$, then $e_{\mathfrak{A}} \otimes \mathscr{L}_\sigma \colon L^1(S) \to \mathfrak{A}$ is totally isolated.*

The theorem immediately follows from Theorem 3.5 by adapting the argument from the proof of Theorem 2.14; or, optionally, by combining Theorem 3.5 with Theorem 2.16 and the fact that $\|\mathscr{L}_\sigma\| = 1$ for all $\sigma \in \widehat{S}_{\mathrm{u}}$.

4. Special cases

In this chapter, we shall deal with three specific semigroup algebras and their homomorphisms. Two of these algebras will be the semigroup algebras of discrete semigroups.

4.1. The algebra $\ell^1(\mathbb{Z}^+)$. Consider $\mathbb{Z}^+$ as a semigroup under addition. The semigroup algebra of $\mathbb{Z}^+$, $\ell^1(\mathbb{Z}^+)$, can be explicitly described as the algebra of all complex-valued, summable sequences $\{f(n)\}_{n\in\mathbb{Z}^+}$, with the norm

$$\|f\| = \sum_{n=0}^{\infty} |f(n)| \quad (f \in \ell^1(\mathbb{Z}^+))$$

and the convolution product

$$(f \star g)(n) = \sum_{m=0}^{n} f(n-m)g(m) \quad (f, g \in \ell^1(\mathbb{Z}^+),\ n \in \mathbb{Z}^+).$$

In preparation for the subsequent discussion, we begin by characterising the semi-characters on $\mathbb{Z}^+$. For each $z \in \overline{\mathbb{D}}$, the mapping $\zeta_z \colon \mathbb{Z}^+ \to \overline{\mathbb{D}}$ given by

$$\zeta_z(n) = z^n \quad (n \in \mathbb{Z}^+)$$

is a semi-character on $\mathbb{Z}^+$. Conversely, every semi-character on $\mathbb{Z}^+$ takes the form ζ_z for some $z \in \overline{\mathbb{D}}$. In other words, $\widehat{\mathbb{Z}^+} = \{\zeta_z \mid z \in \overline{\mathbb{D}}\}$. Here, by convention, $0^0 = 1$, so that ζ_0 is identical with $\chi_{\{0\}}$. It is readily seen that $(\widehat{\mathbb{Z}^+})_{\mathrm{u}} = \{\zeta_z \mid z \in \mathbb{T}\}$ and $(\widehat{\mathbb{Z}^+})_+ = \{\zeta_r \mid 0 \le r \le 1\}$. In particular, the set $(\widehat{\mathbb{Z}^+})_+ \setminus \{1_{\mathbb{Z}^+}\}$, which coincides with $\{\zeta_r \mid 0 \le r < 1\}$, is not empty.

Keeping in line with the standard notation, for each $z \in \overline{\mathbb{D}}$, we abbreviate $\mathscr{L}_{\zeta_z}$ to $\mathscr{L}_z$. Thus

$$\mathscr{L}_z(f) = \sum_{n=0}^{\infty} f(n) z^n \quad (f \in \ell^1(\mathbb{Z}^+)).$$

Treating $\mathbb{Z}^+$ as a subsemigroup of the additive group $\mathbb{Z}$ of integers and bearing in mind Theorems 3.2 and 3.5, Remark 3.6, and Theorem 3.7, we readily obtain the following results.

Theorem 4.1. *We have $\alpha(\ell^1(\mathbb{Z}^+)) = 1$.*

THEOREM 4.2. *Let $\mathfrak{A}$ be a unital normed algebra. If $z \in \mathbb{T}$ and if $H\colon \ell^1(\mathbb{Z}^+) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_z\| < 1$, then $H = e_{\mathfrak{A}} \otimes \mathscr{L}_z$. Moreover, the above assertion fails in general if '$<$' is replaced by '$\leq$'.*

THEOREM 4.3. *Let $\mathfrak{A}$ be a unital normed algebra. If $z \in \mathbb{T}$, then $e_{\mathfrak{A}} \otimes \mathscr{L}_z\colon \ell^1(\mathbb{Z}^+) \to \mathfrak{A}$ is totally isolated.*

To obtain further results, we need a lemma (cf. [13, proof of Theorem 7]).

LEMMA 4.4. *If $0 < r < 1$, then*

$$\lim_{\rho \to r} \sup_{s \geq 0} |r^s - \rho^s| = 0.$$

Proof. Fix $0 < r < 1$. If $0 < \rho < r$, then, as is easily checked, the function $s \mapsto r^s - \rho^s$, $s \geq 0$, is non-negative and attains its maximum at

$$s_{\rho,r} = \frac{\ln\ln\frac{1}{\rho} - \ln\ln\frac{1}{r}}{\ln\frac{1}{\rho} - \ln\frac{1}{r}}.$$

Since $\lim_{\rho \to r} s_{\rho,r} = 1/(\ln(1/r))$, we see that the expression

$$\sup_{s \geq 0} (r^s - \rho^s) = r^{s_{\rho,r}} - \rho^{s_{\rho,r}}$$

converges to 0 as ρ tends to r from the left; in symbols,

$$\lim_{\rho \nearrow r} \sup_{s \geq 0} (r^s - \rho^s) = 0.$$

A similar argument shows that

$$\lim_{\rho \searrow r} \sup_{s \geq 0} (\rho^s - r^s) = 0.$$

The lemma follows. ∎

THEOREM 4.5. *Let $\mathfrak{A}$ be a unital normed algebra. If e is a non-zero idempotent in $\mathfrak{A}$ and if $z \in \mathbb{D}$, then $e \otimes \mathscr{L}_z\colon \ell^1(\mathbb{Z}^+) \to \mathfrak{A}$ is accessible.*

Proof. Write z as $z = ru$, where $r = |z|$ and $|u| = 1$; when $z \neq 0$, we have $u = z/|z|$, and when $z = 0$, we take $u = 1$. Let $\{r_n\}_{n \in \mathbb{N}}$ be a sequence in the open unit interval $(0,1)$ such that $r_n \neq r$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = r$. Then, clearly, $e \otimes \mathscr{L}_{r_n u} \neq e \otimes \mathscr{L}_z$ for all $n \in \mathbb{N}$. Moreover,

$$\|e \otimes \mathscr{L}_z - e \otimes \mathscr{L}_{r_n u}\| = \|e\|\,\|\mathscr{L}_z - \mathscr{L}_{r_n u}\| = \|e\|\,\|\sup_{k \in \mathbb{Z}^+} |r^k - r_n^k| \tag{4.1}$$

for all $n \in \mathbb{N}$. It is now easy to see that

$$\lim_{n \to \infty} \|e \otimes \mathscr{L}_z - e \otimes \mathscr{L}_{r_n u}\| = 0. \tag{4.2}$$

Indeed, if $z \neq 0$, then $0 < r < 1$, and the above equality follows from Lemma 4.4 combined with (4.1). And if $z = 0$, then $r = 0$, so that, necessarily, $\lim_{n\to\infty} r_n = 0$ and

$$\sup_{k \in \mathbb{Z}^+} |r^k - r_n^k| = \sup_{k \in \mathbb{N}} |r^k - r_n^k| = \sup_{k \in \mathbb{N}} r_n^k = r_n$$

for each $n \in \mathbb{N}$, implying further that

$$\lim_{n \to \infty} \sup_{k \in \mathbb{Z}^+} |r^k - r_n^k| = 0;$$

now (4.2) follows by combining the above equality with (4.1). We see that $\{e\otimes\mathscr{L}_{r_n u}\}_{n\in\mathbb{N}}$ has all properties needed to ascertain the accessibility of $e\otimes\mathscr{L}_z$. ∎

We are now in a position to give the following characterisation of continuous homomorphisms from $\ell^1(\mathbb{Z}^+)$.

THEOREM 4.6. *Let $\mathfrak{A}$ be a unital normed algebra. If $H\colon \ell^1(\mathbb{Z}^+)\to\mathfrak{A}$ is a continuous homomorphism, then either:*

(i) *$H=0$ or $H=\mathrm{e}_{\mathfrak{A}}\otimes\mathscr{L}_z$ for some $z\in\mathbb{T}$, in which cases H is totally isolated; or*
(ii) *H is essentially accessible.*

Proof. Corollary 2.20 and Theorem 4.3 guarantee that H is totally isolated when $H=0$ or when $H=\mathrm{e}_{\mathfrak{A}}\otimes\mathscr{L}_z$ for some $z\in\mathbb{T}$. If H is non-zero and not of the form $\mathrm{e}_{\mathfrak{A}}\otimes\mathscr{L}_z$ with $z\in\mathbb{T}$, then either $H=\mathrm{e}_{\mathfrak{A}}\otimes\mathscr{L}_z$ for some $z\in\mathbb{D}$ or H is not scalar. In the first case H is essentially accessible (in fact accessible) by Theorem 4.5, and in the second case H is essentially accessible by Theorem 2.17. ∎

At this stage it will be convenient to return to the question mentioned at the end of Section 2.5 and reveal an ordered AL-algebra admitting accessible zero homomorphisms. Consider $\mathbb{N}$ as a semigroup under addition. The corresponding semigroup algebra $\ell^1(\mathbb{N})$ can be identified with the closed ideal $\{f\in\ell^1(\mathbb{Z}^+)\mid f(0)=0\}$ of $\ell^1(\mathbb{Z}^+)$ and inherits from $\ell^1(\mathbb{Z}^+)$ the structure of an ordered AL-algebra.

THEOREM 4.7. *If $\mathfrak{A}$ is a non-zero normed algebra, then the zero homomorphism from $\ell^1(\mathbb{N})$ to $\mathfrak{A}$ is accessible.*

Proof. For each $a\in\mathfrak{A}$ with $\|a\|\le 1$, define a mapping $H_a\colon\ell^1(\mathbb{N})\to\mathfrak{A}$ by

$$H_a(f)=\sum_{n=1}^{\infty} f(n)a^n \qquad (f=\{f(n)\}_{n\in\mathbb{N}}\in\ell^1(\mathbb{N})).$$

It is readily seen that H_a is a continuous homomorphism such that $\|H_a\|\le\|a\|$ and such that $H_a\neq 0$ whenever $a\neq 0$. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-zero elements of $\mathfrak{A}$ such that $\|a_n\|\le 1$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n=0$. Then $H_{a_n}\neq 0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}\|H_{a_n}\|=0$. This establishes the theorem. ∎

REMARK. Theorems 2.19 and 4.7 immediately imply that $\ell^1(\mathbb{N})$ does not have a bounded approximate identity. But in fact more is true: $\ell^1(\mathbb{N})$ does not have any approximate identity. A simple argument to establish this is based on the observation that $(f\star g)(1)=0$ for all $f,g\in\ell^1(\mathbb{N})$. In light of it,

$$|f(1)|=|f(1)-(e\star f)(1)|\le\|e\star f-f\|_1$$

for all $e,f\in\ell^1(\mathbb{N})$. Thus if $f\in\ell^1(\mathbb{N})$ is such that $f(1)\neq 0$, then there is no $e\in\ell^1(\mathbb{N})$ satisfying $\|f-e\star f\|_1<|f(1)|$. It is clear that this very occurrence prevents $\ell^1(\mathbb{N})$ from having an approximate identity. See also [21, p. 308] for a slightly different, though similar argument.

4.2. The algebra $\ell^1(\mathbb{R}^+)$. Consider $\mathbb{R}^+$ as a semigroup, and $\mathbb{R}$ as a group, both under addition. We shall use the symbol $\mathbb{R}^+_{\mathrm{d}}$ to denote $\mathbb{R}^+$ equipped with the discrete topology;

likewise for $\mathbb{R}_{\mathrm{d}}$. The semigroup algebra of $\mathbb{R}^+_{\mathrm{d}}$, $\ell^1(\mathbb{R}^+)$, can be explicitly described as the algebra of all functions $f\colon \mathbb{R}^+ \to \mathbb{C}$ such that $\sum_{s\geq 0} |f(s)| < \infty$, with the norm

$$\|f\|_1 = \sum_{s\geq 0} |f(s)| \quad (f \in \ell^1(\mathbb{R}^+))$$

and the convolution product

$$(f \star g)(t) = \sum_{0\leq s\leq t} f(s-t)g(t) \quad (f, g \in \ell^1(\mathbb{R}^+),\ t \geq 0).$$

The semi-characters on $\mathbb{R}^+_{\mathrm{d}}$ can be characterised as follows. First, the restriction of every character on $\mathbb{R}_{\mathrm{d}}$ to $\mathbb{R}^+$ is a character on $\mathbb{R}^+_{\mathrm{d}}$. Second, conversely, every character on $\mathbb{R}^+_{\mathrm{d}}$ can be extended in a unique way to a character on $\mathbb{R}_{\mathrm{d}}$ (see, e.g., [36, Proposition 3.5.3]). As a result, the set $(\widehat{\mathbb{R}^+_{\mathrm{d}}})_{\mathrm{u}}$ can be identified with the dual group $\widehat{\mathbb{R}_{\mathrm{d}}}$ of $\mathbb{R}_{\mathrm{d}}$. The group $\widehat{\mathbb{R}_{\mathrm{d}}}$ is usually referred to as the *Bohr compactification* of $\mathbb{R}$, and is denoted by $b\mathbb{R}$ (see, e.g., [32, Sect. 4.7] or [72, Sect. 1.8]). Mindful of this, we can put the point just made as follows: $(\widehat{\mathbb{R}^+_{\mathrm{d}}})_{\mathrm{u}}$ is naturally identifiable with $b\mathbb{R}$. Hereafter we shall use the notation $b\mathbb{R}$ to mean $(\widehat{\mathbb{R}^+_{\mathrm{d}}})_{\mathrm{u}}$.

A simple argument shows that $\chi_{\{0\}}$, viewed as a function on $\mathbb{R}^+$, is the only semi-character on $\mathbb{R}^+_{\mathrm{d}}$ which attains the value zero at some point in $\mathbb{R}^+$—all other semi-characters on $\mathbb{R}^+_{\mathrm{d}}$ do not vanish anywhere on $\mathbb{R}^+$ (see, e.g., [36, Example 3.4.15]). It is also immediate that, for each $0 \leq r \leq 1$, the mapping $s \mapsto r^s$, $s \geq 0$, is a semi-character in $(\widehat{\mathbb{R}^+_{\mathrm{d}}})_+$. Here, we retain the convention that $0^0 = 1$, so that the function $s \mapsto 0^s$ is identical with $\chi_{\{0\}}$. A more involved fact is that, conversely, every semi-character in $(\widehat{\mathbb{R}^+_{\mathrm{d}}})_+$ is of the form $s \mapsto r^s$ for some $0 \leq r \leq 1$ (see, e.g., the proof of Proposition 3.5.11 in [36]).

Given $0 \leq r \leq 1$ and $\chi \in b\mathbb{R}$, let $r \diamond \chi\colon \widehat{\mathbb{R}^+_{\mathrm{d}}} \to \overline{\mathbb{D}}$ be the mapping defined by

$$(r \diamond \chi)(s) = r^s \chi(s) \quad (s \in \mathbb{R}^+).$$

For each $0 \leq r \leq 1$ and each $\chi \in b\mathbb{R}$, $r \diamond \chi$ is a semi-character on $\mathbb{R}^+_{\mathrm{d}}$. It is clear that $r \diamond 1_{\mathbb{R}^+}$ is the same as the mapping $s \mapsto r^s$ for each $0 < r \leq 1$. Accordingly, we can restate the characterisation of $(\widehat{\mathbb{R}^+_{\mathrm{d}}})_+$ given above as

$$(\widehat{\mathbb{R}^+_{\mathrm{d}}})_+ = \{r \diamond 1_{\mathbb{R}^+} \mid 0 \leq r \leq 1\}.$$

In particular, the set $(\widehat{\mathbb{R}^+_{\mathrm{d}}})_+ \setminus \{1_{\mathbb{R}^+}\}$ is not vacuous.

If ζ is a semi-character on $\mathbb{R}^+_{\mathrm{d}}$ different from $\chi_{\{0\}}$, then, given that ζ does not vanish anywhere on $\mathbb{R}^+$, ζ can be uniquely written as $u|\zeta|$, where $u \in b\mathbb{R}$ and $|\zeta| \in (\widehat{\mathbb{R}^+_{\mathrm{d}}})_+$. Consequently,

$$\widehat{\mathbb{R}^+_{\mathrm{d}}} \setminus \{\chi_{\{0\}}\} = \{r \diamond \chi \mid 0 < r \leq 1,\ \chi \in b\mathbb{R}\}.$$

This leads to the following representation of $\widehat{\mathbb{R}^+_{\mathrm{d}}}$:

$$\widehat{\mathbb{R}^+_{\mathrm{d}}} = \{\chi_{\{0\}}\} \cup \{r \diamond \chi \mid 0 < r \leq 1,\ \chi \in b\mathbb{R}\}.$$

In the literature, $\widehat{\mathbb{R}^+_{\mathrm{d}}}$ is often referred to as the *big disc over* $b\mathbb{R}$ or simply the big disc [7], [36], [84]. The "centre" of the big disc is $\chi_{\{0\}}$, its "circumference" is $b\mathbb{R}$, and its "radius" is equal to 1.

If $\zeta \in \widehat{\mathbb{R}_{\mathrm{d}}^{+}}$, then the functional $\mathscr{L}_\zeta$ is explicitly given by

$$\mathscr{L}_\zeta(f) = \sum_{s \geq 0} f(s)\zeta(s) \quad (f \in \ell^1(\mathbb{R}^+)).$$

Treating $\mathbb{R}_{\mathrm{d}}^+$ is a subsemigroup of $\mathbb{R}_{\mathrm{d}}$ and bearing in mind Theorems 3.2 and 3.5, Remark 3.6, and Theorem 3.7, we immediately deduce the following results.

THEOREM 4.8. *We have* $\alpha(\ell^1(\mathbb{R}^+)) = 1$.

THEOREM 4.9. *Let* $\mathfrak{A}$ *be a unital normed algebra. If* $\chi \in b\mathbb{R}$ *and if* $H \colon \ell^1(\mathbb{R}^+) \to \mathfrak{A}$ *is a continuous homomorphism such that* $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\chi\| < 1$, *then* $H = e_{\mathfrak{A}} \otimes \mathscr{L}_\chi$. *Moreover, the above assertion fails in general if* $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\chi\| < 1$ *is replaced by* $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_\chi\| \leq 1$.

THEOREM 4.10. *Let* $\mathfrak{A}$ *be a unital normed algebra. If* $\chi \in b\mathbb{R}$, *then* $e_{\mathfrak{A}} \otimes \mathscr{L}_\chi \colon \ell^1(\mathbb{R}^+) \to \mathfrak{A}$ *is totally isolated.*

Theorems 4.9 and 4.10 are naturally complemented by the two results presented next.

THEOREM 4.11. *Let* $\mathfrak{A}$ *be a unital normed algebra. If* $H \colon \ell^1(\mathbb{R}^+) \to \mathfrak{A}$ *is a continuous homomorphism such that* $\|H - e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}}\| < 1$, *then* $H = e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}}$. *Moreover, the above assertion fails in general if '$<$' is replaced by '$\leq$'.*

Proof. Let $r := \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}}\|$. By assumption, $r < 1$. If $s > 0$, then $\mathscr{L}_{\chi\{0\}}(\delta_s) = 0$ and $\|\delta_s\|_1 = 1$, and so

$$\|H(\delta_s)\| = \|(H - e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}})(\delta_s)\| \leq \|H - e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}}\| \, \|\delta_s\|_1 = r.$$

Now, for each $s > 0$ and each $n \in \mathbb{N}$,

$$H(\delta_s) = H(\delta_{s/n} \star \cdots \star \delta_{s/n}) = (H(\delta_{s/n}))^n,$$

and further, by the inequality in the previous sentence,

$$\|H(\delta_s)\| \leq \|H(\delta_{s/n})\|^n \leq r^n.$$

By letting $n \to \infty$, we find that $H(\delta_s) = 0$ for each $s > 0$. It follows that $H = e \otimes \mathscr{L}_{\chi\{0\}}$, where $e = H(\delta_0)$. Since δ_0 is idempotent, so too is its homomorphic image e. Now, as $\|\mathscr{L}_{\chi\{0\}}\| = \|\chi\{0\}\|_\infty = 1$, we can apply Lemma 2.15 to conclude that $e = e_{\mathfrak{A}}$. Hence, at once, $H = e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}}$.

Finally, to see that '$<$' in the hypothesis of the theorem cannot be replaced by '$\leq$', note that $e_{\mathfrak{A}} \otimes \mathscr{L}_{1_{\mathbb{R}^+}} \neq e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}}$ and

$$\|e_{\mathfrak{A}} \otimes \mathscr{L}_{1_{\mathbb{R}^+}} - e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}}\| = \|1_{\mathbb{R}^+} - \chi\{0\}\|_\infty = 1. \blacksquare$$

Using Theorem 4.11 and adapting the argument from the proof of Theorem 2.14 (or alternatively, but a bit convolutedly, using Theorems 4.11 and 2.16 and the fact that $\|\mathscr{L}_{\chi\{0\}}\| = 1$), we obtain the following result.

THEOREM 4.12. *Let* $\mathfrak{A}$ *be a unital normed algebra. Then* $e_{\mathfrak{A}} \otimes \mathscr{L}_{\chi\{0\}} \colon \ell^1(\mathbb{R}^+) \to \mathfrak{A}$ *is totally isolated.*

Our next result reveals accessible scalar homomorphisms from $\ell^1(\mathbb{R}^+)$.

THEOREM 4.13. *Let* $\mathfrak{A}$ *be a unital normed algebra. If* e *is a non-zero idempotent in* $\mathfrak{A}$, *if* $0 < r < 1$, *and if* $\chi \in b\mathbb{R}$, *then* $e \otimes \mathscr{L}_{r \diamond \chi} \colon \ell^1(\mathbb{R}^+) \to \mathfrak{A}$ *is accessible.*

Proof. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $r_n \neq r$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} r_n = r$. Clearly, $e \otimes \mathscr{L}_{r_n\diamond\chi} \neq e \otimes \mathscr{L}_{r\diamond\chi}$ for all $n \in \mathbb{N}$. Moreover, since

$$\|e \otimes \mathscr{L}_{r\diamond\chi} - e \otimes \mathscr{L}_{r_n\diamond\chi}\| = \|e\|\|\mathscr{L}_{r\diamond\chi} - \mathscr{L}_{r_n\diamond\chi}\| = \|e\|\| \sup_{s\geq 0} |r^s - r_n^s|$$

for all $n \in \mathbb{N}$, it follows from Lemma 4.4 that $\lim_{n\to\infty} \|e \otimes \mathscr{L}_{r\diamond\chi} - e \otimes \mathscr{L}_{r_n\diamond\chi}\| = 0$. Thus $\{e \otimes \mathscr{L}_{r_n\diamond\chi}\}_{n\in\mathbb{N}}$ has both of the properties needed to guarantee the accessibility of $e \otimes \mathscr{L}_{r\diamond\chi}$. ∎

We can now give the following characterisation of continuous homomorphisms from $\ell^1(\mathbb{R}^+)$.

THEOREM 4.14. *Let $\mathfrak{A}$ be a unital normed algebra. If $H\colon \ell^1(\mathbb{R}^+) \to \mathfrak{A}$ is a continuous homomorphism, then either:*

(i) *$H = 0$, or $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_\chi$ for some $\chi \in b\mathbb{R}$, or $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_{\chi_{\{0\}}}$, in each of which cases H is totally isolated; or*
(ii) *H is essentially accessible.*

Proof. Corollary 2.20 and Theorems 4.10 and 4.12 guarantee that H is totally isolated when $H = 0$, or when $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_\chi$ for some $\chi \in b\mathbb{R}$, or when $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_{\chi_{\{0\}}}$. If H is non-zero, different from $\mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_\chi$ for every $\chi \in b\mathbb{R}$, and different from $\mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_{\chi_{\{0\}}}$, then either $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_{r\diamond\chi}$ for some $0 < r < 1$ and some $\chi \in b\mathbb{R}$, or H is not scalar. In the first case H is essentially accessible (in fact accessible) by Theorem 4.13, and in the second case H is essentially accessible by Theorem 2.17. ∎

4.3. The algebra $L^1(\mathbb{R}^+)$. Consider $\mathbb{R}^+$ with its usual topology. With $\mathbb{R}^+$ viewed as a (Lebesgue) measurable subsemigroup of $\mathbb{R}$, the semigroup algebra of $\mathbb{R}^+$ is the same as the algebra $L^1(\mathbb{R}^+)$ introduced in the Introduction. Let us briefly discuss the semi-characters on $\mathbb{R}^+$. For each $z \in \overline{\mathbb{C}^+}$, the mapping $\epsilon_z\colon \mathbb{R}^+ \to \overline{\mathbb{D}}$ given by

$$\epsilon_z(s) = \mathrm{e}^{-zs} \qquad (s \in \mathbb{R}^+)$$

is a semi-character on $\mathbb{R}^+$. Conversely, every semi-character on $\mathbb{R}^+$ takes the form ϵ_z for some $z \in \overline{\mathbb{C}^+}$. In other words, $\widehat{\mathbb{R}^+} = \{\epsilon_z \mid z \in \overline{\mathbb{C}^+}\}$. It is readily seen that $(\widehat{\mathbb{R}^+})_{\mathrm{u}} = \{\epsilon_z \mid z \in \mathrm{i}\mathbb{R}\}$ and $(\widehat{\mathbb{R}^+})_+ = \{\epsilon_r \mid 0 \leq r < \infty\}$. In particular, the set $(\widehat{\mathbb{R}^+})_+ \setminus \{1_{\mathbb{R}^+}\}$, which coincides with $\{\epsilon_r \mid 0 < r < \infty\}$, is not empty.

Henceforth, for each $z \in \overline{\mathbb{C}^+}$, we shall abbreviate $\mathscr{L}_{\epsilon_z}$ to $\mathscr{L}_z$, so that

$$\mathscr{L}_z(f) = \int_0^\infty f(s)\mathrm{e}^{-zs}\,\mathrm{d}s \qquad (f \in L^1(\mathbb{R}^+)).$$

Treating $\mathbb{R}^+$ as a subsemigroup of $\mathbb{R}$ and bearing in mind Theorems 3.2 and 3.5, Remark 3.6, and Theorem 3.7, we readily obtain the following results.

THEOREM 4.15. *We have $\alpha(L^1(\mathbb{R}^+)) = 1$.*

THEOREM 4.16. *Let $\mathfrak{A}$ be a unital normed algebra. If $z \in \mathrm{i}\mathbb{R}$ and if $H\colon L^1(\mathbb{R}^+) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_z\| < 1$, then $H = e_{\mathfrak{A}} \otimes \mathscr{L}_z$. Moreover, the above assertion fails in general if '$<$' is replaced by '$\leq$'.*

THEOREM 4.17. *Let $\mathfrak{A}$ be a unital normed algebra. If $z \in \mathrm{i}\mathbb{R}$, then $e_{\mathfrak{A}} \otimes \mathscr{L}_z\colon L^1(\mathbb{R}^+) \to \mathfrak{A}$ is totally isolated.*

To continue, we shall need a lemma.

LEMMA 4.18. *If $z \in \mathbb{C}^+$, then*

$$\lim_{\substack{\zeta \to z \\ \operatorname{Im}\zeta = \operatorname{Im} z}} \sup_{s \geq 0} |\mathrm{e}^{-zs} - \mathrm{e}^{-\zeta s}| = 0.$$

Proof. If $\lambda > 0$ and if $\omega \in \mathbb{R}$, then

$$\lim_{\substack{\nu \to \lambda \\ \nu > 0}} \sup_{s \geq 0} |\mathrm{e}^{-(\lambda + \mathrm{i}\omega)s} - \mathrm{e}^{-(\nu + \mathrm{i}\omega)s}| = \lim_{\substack{\nu \to \lambda \\ \nu > 0}} \sup_{s \geq 0} |\mathrm{e}^{-\lambda s} - \mathrm{e}^{-\nu s}| = 0$$

by Lemma 4.4. ∎

THEOREM 4.19. *Let $\mathfrak{A}$ be a normed algebra. If e is a non-zero idempotent in $\mathfrak{A}$ and if $z \in \mathbb{C}^+$, then $e \otimes \mathscr{L}_z : L^1(\mathbb{R}^+) \to \mathfrak{A}$ is accessible.*

Proof. Let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{C}^+$ such that $z_n \neq z$ and $\operatorname{Im} z_n = \operatorname{Im} z$ for each $n \in \mathbb{N}$ and such that $\lim_{n\to\infty} z_n = z$. Clearly, $e \otimes \mathscr{L}_{z_n} \neq e \otimes \mathscr{L}_z$ for all $n \in \mathbb{N}$. Moreover, since

$$\|e \otimes \mathscr{L}_z - e \otimes \mathscr{L}_{z_n}\| = \|e\|\, \|\mathscr{L}_z - \mathscr{L}_{z_n}\| = \|e\| \sup_{s \geq 0} |\mathrm{e}^{-zs} - \mathrm{e}^{-z_n s}|$$

for all $n \in \mathbb{N}$, it follows from Lemma 4.18 that $\lim_{n\to\infty} \|e \otimes \mathscr{L}_z - e \otimes \mathscr{L}_{z_n}\| = 0$. Thus $\{e \otimes \mathscr{L}_{z_n}\}_{n \in \mathbb{N}}$ has both of the properties required to assert the accessibility of $e \otimes \mathscr{L}_z$. ∎

We can now present the following characterisation of continuous homomorphisms from $L^1(\mathbb{R}^+)$.

THEOREM 4.20. *Let $\mathfrak{A}$ be a unital normed algebra. If $H : L^1(\mathbb{R}^+) \to \mathfrak{A}$ is a continuous homomorphism, then either:*

(i) *$H = 0$ or $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_z$ for some $z \in \mathrm{i}\mathbb{R}$, in which cases H is totally isolated; or*
(ii) *H is essentially accessible.*

Proof. Corollary 2.20 and Theorem 4.17 guarantee that H is totally isolated when $H = 0$ or when $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_z$ for some $z \in \mathrm{i}\mathbb{R}$. If H is non-zero and not of the form $\mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_z$ with $z \in \mathrm{i}\mathbb{R}$, then either $H = \mathrm{e}_{\mathfrak{A}} \otimes \mathscr{L}_w$ for some $w \in \mathbb{C}^+$ or H is not scalar. In the first case H is essentially accessible (in fact accessible) by Theorem 4.19, and in the second case H is essentially accessible by Theorem 2.17. ∎

4.4. Complement. Let $\mathfrak{A}$ be a unital normed algebra, and let e be a non-zero idempotent in $\mathfrak{A}$. In the course of proving Theorem 4.19, it was shown that, for each $z \in \mathbb{C}^+$, $e \otimes \mathscr{L}_z$ can be approximated in norm by homomorphisms of the form $e \otimes \mathscr{L}_w$, $w \in \mathbb{C}^+ \setminus \{z\}$. As a complement to this finding, we now show that there exist a normed algebra $\mathfrak{B}$ and an isometric homomorphism $I : \mathfrak{A} \to \mathfrak{B}$ such that each $I \circ (e \otimes \mathscr{L}_z)$, $z \in \mathbb{C}^+$, can be approximated by homomorphisms different from any of the $I \circ (e \otimes \mathscr{L}_w)$, $w \in \mathbb{C}^+ \setminus \{z\}$.

EXAMPLE 4.21. Let $\mathfrak{A} \oplus \mathfrak{A}$ be the direct sum of two copies of $\mathfrak{A}$, endowed with the norm

$$\|(x, y)\| = \max(\|x\|, \|y\|) \quad (x, y \in \mathfrak{A}).$$

Let $\mathfrak{B} = \mathscr{L}(\mathfrak{A} \oplus \mathfrak{A})$, and let $I : \mathfrak{A} \to \mathfrak{B}$ be the isometric homomorphism defined by

$$I(w)(x, y) = (wx, wy) \quad (w, x, y \in \mathfrak{A}).$$

Fix $z \in \mathbb{C}^+$, and let $\lambda := \operatorname{Re} z$ and $\omega := \operatorname{Im} z$. Pick two sequences of positive numbers $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\nu_n\}_{n\in\mathbb{N}}$ converging to λ such that $\mu_n \neq \nu_n$ for all $n \in \mathbb{N}$. For each $f \in L^1(\mathbb{R}^+)$ and each $n \in \mathbb{N}$, define a homomorphism G_n in $\operatorname{Hom}(L^1(\mathbb{R}^+), \mathfrak{B})$ by

$$G_n(f)(x,y) = ((e \otimes \mathscr{L}_{\mu_n+\mathrm{i}\omega})(f)x, (e \otimes \mathscr{L}_{\nu_n+\mathrm{i}\omega})(f)y) \quad (f \in L^1(\mathbb{R}^+),\ x, y \in \mathfrak{A}).$$

In matrix form,

$$G_n(f)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathscr{L}_{\mu_n+\mathrm{i}\omega}(f)e & 0 \\ 0 & \mathscr{L}_{\nu_n+\mathrm{i}\omega}(f)e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Taking into account that, for every $w \in \mathbb{C}^+$, $(I \circ (e \otimes \mathscr{L}_w))(f)$ may be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \mathscr{L}_w(f)e & 0 \\ 0 & \mathscr{L}_w(f)e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

it is clear that each G_n is different from any of the $I \circ (e \otimes \mathscr{L}_w)$, $w \in \mathbb{C}^+$. Since

$$\|I \circ (e \otimes \mathscr{L}_z) - G_n\| \le \max(\|\mathscr{L}_z - \mathscr{L}_{\mu_n+\mathrm{i}a}\|, \|\mathscr{L}_z - \mathscr{L}_{\nu_n+\mathrm{i}a}\|)\|e\|$$

for all $n \in \mathbb{N}$, it follows from Lemma 4.18 that $\lim_{n\to\infty} \|I \circ (p \otimes \mathscr{L}_z) - G_n\| = 0$.

Example 4.22. We retain the notation from the previous example. With $z \in \mathbb{C}^+$ fixed, for each $\epsilon > 0$, let $\{\mathscr{S}_\epsilon(t)\}_{t\ge 0}$ be the one-parameter semigroup on $\mathfrak{B}$ defined by

$$\mathscr{S}_\epsilon(t)\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} \mathrm{e}^{-zt}p & \epsilon t\mathrm{e}^{-zt}p \\ 0 & \mathrm{e}^{-zt}p \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (t \ge 0,\ x, y \in \mathfrak{A}).$$

Clearly, $\sup_{t\ge 0} |\mathrm{e}^{-zt}| = 1$. Also, it is easy to see that the function $t \mapsto t\mathrm{e}^{-\lambda t}$, $t \ge 0$, attains its maximum at $t = \lambda^{-1}$, so that

$$\sup_{t\ge 0} t|\mathrm{e}^{-zt}| = \sup_{t\ge 0} t\mathrm{e}^{-\lambda t} = \lambda^{-1}\mathrm{e}^{-1}. \tag{4.3}$$

Consequently, $\{\mathscr{S}_\epsilon(t)\}_{t\ge 0}$ is uniformly bounded—that is, $\sup_{t\ge 0} \|\mathscr{S}_\epsilon(t)\| < \infty$. We may then define a homomorphism G_ϵ in $\operatorname{Hom}(L^1(\mathbb{R}^+), \mathfrak{B})$ by

$$G_\epsilon(f) = \int_0^\infty \mathscr{S}_\epsilon(t) f(t)\,\mathrm{d}t \quad (f \in L^1(\mathbb{R}^+)).$$

In explicit form,

$$G_\epsilon(f)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathscr{L}_z(f)e & \epsilon\mathscr{Q}_z(f)e \\ 0 & \mathscr{L}_z(f)e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (f \in L^1(\mathbb{R}^+),\ x, y \in \mathfrak{A}),$$

where $\mathscr{Q}_z$ is the linear functional on $L^1(\mathbb{R}^+)$ defined by

$$\mathscr{Q}_z(f) = \int_0^\infty t\mathrm{e}^{-zt} f(t)\,\mathrm{d}t \quad (f \in L^1(\mathbb{R}^+)).$$

In view of (4.3), $\mathscr{Q}_z$ is bounded with norm $\lambda^{-1}\mathrm{e}^{-1}$. For each $f \in L^1(\mathbb{R}^+)$, let $Q_\epsilon(f)$ be the member of $\mathscr{L}(\mathfrak{B})$ given by

$$Q_\epsilon(f)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & \epsilon\mathscr{Q}_z(f)e \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (x, y \in \mathfrak{A}).$$

Clearly, $Q_\epsilon(f)$ is nilpotent with index 2, that is, $(Q_\epsilon(f))^2 = 0$, and we have

$$G_\epsilon(f) = (I \circ (e \otimes \mathscr{L}_z))(f) + Q_\epsilon(f).$$

Since $(I \circ (e \otimes \mathscr{L}_z))(f)$ is of scalar type and commutes with $Q_\epsilon(f)$, it follows that $G_\epsilon(f)$ is a spectral operator in the sense of Dunford [28]. $Q_\epsilon(f)$ is the uniquely defined, non-zero nilpotent part of $G_\epsilon(f)$, and so $G_\epsilon(f)$ is not of scalar type. On the other hand,

$$\|(I \circ (e \otimes \mathscr{L}_z))(f) - G_\epsilon(f)\| \le \epsilon \|\mathscr{Q}_z\| \, \|f\|_1 \|e\|$$

for each $f \in L^1(\mathbb{R}^+)$, and so $\lim_{\epsilon \to 0} \|I \circ (e \otimes \mathscr{L}_z) - G_\epsilon\| = 0$. We conclude that $I \circ (e \otimes \mathscr{L}_z)$ can be approximated by homomorphisms whose values are spectral operators not of scalar type.

5. Group algebras

All the semigroup algebras considered in the previous chapter have α-number equal to 1. As it turns out, an ample supply of ordered AL-algebras with α-number greater than 1 is furnished by the group algebras of non-zero, locally compact Abelian groups. This chapter is concerned with algebras of that kind, focusing on isolability properties of respective algebra homomorphisms.

5.1. Examples. We start with a few examples concerning the group algebras of specific finite Abelian groups. These examples will not only instantly reveal ordered AL-algebras with α-number greater than 1, but will also prepare us for a more systematic approach towards calculation of the α-numbers of various group algebras.

Below, for $n \in \mathbb{N}$, $\mathbb{Z}_n$ be will viewed as the additive group of integers modulo n.

Example 5.1. Let $\mathfrak{A}$ be a unital normed algebra, and let $H \colon \ell^1(\mathbb{Z}_2) \to \mathfrak{A}$ be a homomorphism. If $f \in \ell^1(\mathbb{Z}_2)$, then $f = f(0)\delta_0 + f(1)\delta_1$, and further $H(f) = f(0)h_0 + f(1)h_1$, where $h_0 = H(\delta_0)$ and $h_1 = H(\delta_1)$. The fundamental homomorphism $e_\mathfrak{A} \otimes l \colon \ell^1(\mathbb{Z}_2) \to \mathfrak{A}$ takes the form

$$(e_\mathfrak{A} \otimes l)(f) = (f(0) + f(1))e_\mathfrak{A} \qquad (f \in \ell^1(\mathbb{Z}_2)).$$

It follows that

$$\begin{aligned} \|H - e_\mathfrak{A} \otimes l\| &= \sup_{|f(0)|+|f(1)|=1} \|f(0)(h_0 - e_\mathfrak{A}) + f(1)(h_1 - e_\mathfrak{A})\| \\ &= \max(\|h_0 - e_\mathfrak{A}\|, \|h_1 - e_\mathfrak{A}\|). \end{aligned} \tag{5.1}$$

Suppose henceforth that $\mathfrak{A}$ possesses merely trivial idempotents and that H is non-zero. The identities

$$\delta_0 * \delta_0 = \delta_0, \qquad \delta_1 * \delta_1 = \delta_0, \quad \text{and} \quad \delta_0 * \delta_1 = \delta_1$$

imply that

$$h_0^2 = h_0, \qquad h_1^2 = h_0, \quad \text{and} \quad h_0 h_1 = h_1. \tag{5.2}$$

The first identity of (5.2) proclaims that h_0 is idempotent, so $h_0 = e_\mathfrak{A}$ or $h_0 = 0$. By the third identity of (5.2), if $h_0 = 0$, then $h_1 = 0$, and hence $H = 0$, which is excluded by assumption. Therefore, $h_0 = e_\mathfrak{A}$. Now the second identity of (5.2) becomes $h_1^2 = e_\mathfrak{A}$. By applying the lemma below, we find that $h_1 = e_\mathfrak{A}$ or $h_1 = -e_\mathfrak{A}$.

LEMMA 5.2. *Let* $\mathfrak{A} \in \mathscr{A}_{\mathrm{wni}}$ *and let* $x \in \mathfrak{A}$ *be such that* $x^2 = e_{\mathfrak{A}}$. *Then either* $x = e_{\mathfrak{A}}$ *or* $x = -e_{\mathfrak{A}}$.

Proof. Assume that $x \neq e_{\mathfrak{A}}$ and $x \neq -e_{\mathfrak{A}}$. Set $y = (e_{\mathfrak{A}} - x)/2$. Then $y \neq 0$ and $y \neq e_{\mathfrak{A}}$ while $y^2 = y$. But this contradicts the assumption that $\mathfrak{A}$ does not have a non-trivial idempotent. The proof is complete. ∎

Returning to our example, we see that H takes the form

$$H(f) = (f(0) \pm f(1))e_{\mathfrak{A}} \quad (f \in \ell^1(\mathbb{Z}_2)).$$

If H is different from $e_{\mathfrak{A}} \otimes l$, then, necessarily,

$$H(f) = (f(0) - f(1))e_{\mathfrak{A}} \quad (f \in \ell^1(\mathbb{Z}_2)),$$

and now (5.1) implies that $\|H - e_{\mathfrak{A}} \otimes l\| = 2$. We conclude that $\alpha(\ell^1(\mathbb{Z}_2)) = 2$.

EXAMPLE 5.3. The Klein four-group K_4, also denoted by V or V_4 (for *Vierergruppe*), is the abstract Abelian group with four elements s_1, s_2, s_3, s_4 obeying the multiplication table given in Table 1, where s_1 represents the neutral element of the group. It is an elementary fact that K_4 is isomorphic to the direct sum $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [77, p. 58]). Let $\mathfrak{A}$ be

Table 1. Multiplication table for K_4

	s_1	s_2	s_3	s_4
s_1	s_1	s_2	s_3	s_4
s_2	s_2	s_1	s_4	s_3
s_3	s_3	s_4	s_1	s_2
s_4	s_4	s_3	s_2	s_1

a unital normed algebra, and let $H\colon \ell(K_4) \to \mathfrak{A}$ be a homomorphism. If $f \in \ell^1(K_4)$, then $f = \sum_{i=1}^4 f(s_i)\delta_{s_i}$, and further $H(f) = \sum_{i=1}^4 f(s_i)h_i$, where $h_i = H(\delta_{s_i})$ for $i = 1, 2, 3, 4$. The fundamental homomorphism $e_{\mathfrak{A}} \otimes l\colon \ell^1(K_4) \to \mathfrak{A}$ takes the form

$$(e_{\mathfrak{A}} \otimes l)(f) = \Big(\sum_{i=1}^{4} f(s_i)\Big)e_{\mathfrak{A}} \quad (f \in \ell^1(K_4)),$$

and we have

$$\|H - e_{\mathfrak{A}} \otimes l\| = \max_{1 \le i \le 4} \|h_i - e_{\mathfrak{A}}\|. \tag{5.3}$$

Suppose henceforth that $\mathfrak{A} \in \mathscr{A}_{\mathrm{wni}}$ and that $H \neq 0$. Since

$$\delta_{s_1} * \delta_{s_1} = \delta_{s_1} \quad \text{and} \quad \delta_{s_1} * \delta_{s_j} = \delta_{s_j} \quad (j = 2, 3, 4),$$

we have

$$h_1^2 = h_1 \quad \text{and} \quad h_1 h_j = h_j \quad (j = 2, 3, 4). \tag{5.4}$$

By a natural extension of the argument used in the previous example, these identities imply that $h_1 = e_{\mathfrak{A}}$. Since

$$\delta_{s_j} * \delta_{s_j} = \delta_{s_1} \quad (j = 2, 3, 4),$$

we have

$$h_j^2 = e_{\mathfrak{A}} \quad (j = 2, 3, 4), \tag{5.5}$$

and hence, by Lemma 5.2,

$$h_j = e_{\mathfrak{A}} \quad \text{or} \quad h_j = -e_{\mathfrak{A}} \quad (j = 2, 3, 4).$$

Now, in view of (5.3), it is clear that $\|H - e_{\mathfrak{A}} \otimes l\| \leq 2$, and hence that

$$\alpha(\ell(K_4)) \leq 2. \tag{5.6}$$

Next, define $H\colon \ell^1(K_4) \to \mathbb{C}$ by

$$H(f) = f(s_1) - f(s_2) - f(s_3) + f(s_4) \quad (f \in \ell^1(K_4)).$$

The corresponding h_i's, defined as previously by $h_i = H(\delta_{s_i})$ for $i = 1, 2, 3, 4$, take the form

$$h_1 = h_4 = 1 \quad \text{and} \quad h_2 = h_3 = -1,$$

and clearly satisfy (5.4), (5.5) with $e_{\mathfrak{A}} = 1$, and

$$h_2 h_3 = h_4, \quad h_2 h_4 = h_3, \quad h_3 h_4 = h_2.$$

Comparing the above relations with the multiplication table of K_4, we see that the mapping $s_i \mapsto h_i$ is a homomorphism from K_4 onto the multiplicative group $\{-1, 1\}$, and hence is a character on K_4. Correspondingly, H is a character on $\ell^1(K_4)$. Since $\max_{1 \leq i \leq 4} |h_i - 1| = 2$, it follows from (5.3) that $\|H - 1 \otimes l\| = 2$. This together with (5.6) yields $\alpha(\ell(K_4)) = 2$.

EXAMPLE 5.4. Every homomorphism H from $\ell^1(\mathbb{Z}_3)$ to a unital normed algebra $\mathfrak{A}$ without non-trivial idempotents is of the form

$$H(f) = f(0)e_{\mathfrak{A}} + f(1)h + f(2)h^2 \quad (f \in \ell^1(\mathbb{Z}_3)), \tag{5.7}$$

where $h \in \mathfrak{A}$ satisfies $h^3 = e_{\mathfrak{A}}$. Note that, if $h = h^2$, then $h^2 = h^3$, and further $h = h^3 = e_{\mathfrak{A}}$, implying that $H = e_{\mathfrak{A}} \otimes l$. Therefore, for $H \neq e_{\mathfrak{A}} \otimes l$, the last of the formulae

$$(e_{\mathfrak{A}} - h)^3 = 3(h^2 - h), \quad (e_{\mathfrak{A}} - h^2)^3 = 3(h - h^2), \quad (h - h^2)^3 = 3(h^2 - h)$$

implies that $\|h - h^2\| \geq \sqrt{3}$. Combining this inequality with the remaining two formulae, we see that both $\|e_{\mathfrak{A}} - h\|$ and $\|e_{\mathfrak{A}} - h^2\|$ are no smaller than $\sqrt{3}$. Since

$$\|H - e_{\mathfrak{A}} \otimes l\| = \max\{\|e_{\mathfrak{A}} - h\|, \|e_{\mathfrak{A}} - h^2\|\}, \tag{5.8}$$

we have

$$\alpha(\ell^1(\mathbb{Z}_3)) \geq \sqrt{3}. \tag{5.9}$$

On the other hand, if we take $\mathfrak{A} = \mathbb{C}$ and set H to be as in (5.7) with $h = \exp(2\pi i/3)$, then $|h - 1| = |h^2 - 1| = \sqrt{3}$, and hence, by (5.8), $\|H - 1 \otimes l\| = \sqrt{3}$. This combined with (5.9) establishes that $\alpha(\ell^1(\mathbb{Z}_3)) = \sqrt{3}$.

5.2. The β-number. The examples given in the previous section show, demonstrably, that determining the α-number of a group algebra can be a tedious and quite involved task. Here we introduce a certain numerical characteristic of an arbitrary non-zero, locally compact Abelian group, and use it to compute α-numbers for a large assortment of group algebras of locally compact Abelian groups (see the end of Section 5.5). The idea behind this characteristic comes partially from the analysis of common elements shared by the examples just mentioned.

From now on, Abelian groups will be written additively, groups of characters and groups of multiplicatively invertible elements of an algebra excepted. The neutral element of an Abelian group will be denoted by 0. If G is an Abelian group, with or without a topology under which G is a locally compact group, then G_{d} will denote G with the discrete topology. The dual group of G_{d}, $\widehat{G_{\mathrm{d}}}$, consists of all characters on G. If G is a locally compact Abelian group, then 1_G is the common neutral element of $\widehat{G}$ and $\widehat{G_{\mathrm{d}}}$. When G is non-zero, both $\widehat{G}\setminus\{1_G\}$ and $\widehat{G_{\mathrm{d}}}\setminus\{1_G\}$ are non-empty.

For a non-zero, locally compact Abelian group G, we let

$$\beta(G) := \inf_{\chi\in\widehat{G}\setminus\{1_G\}} \sup_{s\in G} |\chi(s)-1|.$$

We shall refer to $\beta(G)$ as the *β-number* of G. Note that $\beta(G_{\mathrm{d}}) \le \beta(G)$, which immediately results from the relation $\widehat{G}\subset\widehat{G_{\mathrm{d}}}$.

We now give an early indication of the range of values assumed by β-numbers.

Theorem 5.5. *If G is a non-zero, locally compact Abelian group, then $\sqrt{3}\le\beta(G)\le 2$.*

Remark. As it turns out, both bounds appearing in the above statement are tight: we have $\beta(\mathbb{Z}_3)=\sqrt{3}$ and $\beta(\mathbb{Z}_2)=2$; see Corollary 5.18 below.

Proof of Theorem 5.5. The upper bound is clear, since every member of $\widehat{G}$ is unitary.

To establish the lower bound, we argue by contradiction. Assume that there exists a non-zero, locally compact Abelian group G for which $\beta(G)<\sqrt{3}$. One can then choose $\chi\in\widehat{G}\setminus\{1_G\}$ such that the image $\chi(G)$ of G by χ is contained in $\Gamma_{\epsilon_0} := \{z\in\mathbb{T} \mid |z-1|\le \sqrt{3}-\epsilon_0\}$ for some $0<\epsilon_0<\sqrt{3}$. Note that

$$\Gamma_{\epsilon_0} = \left\{ \mathrm{e}^{i\omega} \;\middle|\; |\omega| \le \frac{2\pi}{3}-\epsilon \right\},$$

where $\epsilon := 2\pi/3 - 2\arcsin((\sqrt{3}-\epsilon_0)/2)$ is such that $0<\epsilon<2\pi/3$. Let

$$\omega_0 := \sup\{|\omega| \mid \mathrm{e}^{i\omega}\in\chi(G),\ -\pi<\omega\le\pi\}.$$

Since $\chi(G)$ is symmetric in the sense that $\mathrm{e}^{i\omega}\in\chi(G)$ implies $\mathrm{e}^{-i\omega}\in\chi(G)$, and since $\chi\ne 1_G$, we have

$$0<\omega_0 = \sup\{\omega \mid \mathrm{e}^{i\omega}\in\chi(G),\ 0\le\omega\le\pi\}.$$

Using the second expression for Γ_{ϵ_0} together with the assumption that $\chi(G)\subset\Gamma_{\epsilon_0}$, we conclude that $\omega_0\le 2\pi/3-\epsilon$.

For each $z\in\mathbb{C}\setminus\{0\}$, let $\operatorname{Arg} z$ denote the principal argument of z defined as the unique $0\le\theta<2\pi$ such that $z=|z|\mathrm{e}^{\mathrm{i}\theta}$. It is clear that

$$\omega_0 = \sup\{\operatorname{Arg} z \mid z\in\chi(G)\cap\mathbb{H}\}.$$

We now consider two cases (see Figure 1).

Case 1. Suppose first that $\omega_0\le\pi/2$, and let $s\in G$ be such that $2\omega_0/3\le\operatorname{Arg}\chi(s)\le\omega_0$. Then

$$\omega_0 < \frac{4\omega_0}{3} \le 2\operatorname{Arg}\chi(s) \le 2\omega_0 \le \pi.$$

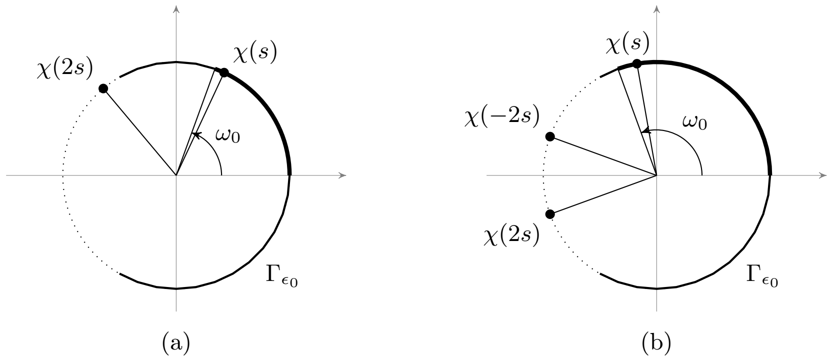

Fig. 1. The set $\chi(G) \cap \mathbb{H}$ is contained in the thick-filled arc. In panel (a), $\omega_0 \le \pi/2$, and in panel (b), $\pi/2 < \omega_0 \le 2\pi/3 - \epsilon$.

The inequality $2 \operatorname{Arg} \chi(s) \le \pi$ ensures that $2 \operatorname{Arg} \chi(s)$ equals $\operatorname{Arg} (\chi(s))^2$, which in turn equals $\operatorname{Arg} \chi(2s)$. The same inequality written as $\operatorname{Arg} \chi(2s) \le \pi$ amounts to saying that $\chi(2s)$ is a member of $\chi(G) \cap \mathbb{H}$. But, since $\omega_0 < 2 \operatorname{Arg} \chi(s) = \operatorname{Arg} \chi(2s)$ and since no element of $\chi(G) \cap \mathbb{H}$ may have the principal argument greater than ω_0, we obtain a contradiction.

Case 2. Suppose next that $\omega_0 = \pi/2 + \delta$ for some $\delta > 0$. Let $s \in G$ be such that $\omega_0 - \delta/2 \le \operatorname{Arg} \chi(s) \le \omega_0$. Then, clearly,

$$2\omega_0 - \delta \le 2 \operatorname{Arg} \chi(s) \le 2\omega_0. \tag{5.10}$$

Note that $\omega_0 \le 2\pi/3 - \epsilon$ implies that $2\omega_0 < 2\pi$. Therefore $2 \operatorname{Arg} \chi(s) < 2\pi$, and, by an argument as above, $2 \operatorname{Arg} \chi(s) = \operatorname{Arg} \chi(2s)$. This combined with $\operatorname{Arg} \chi(-2s) = 2\pi - \operatorname{Arg} \chi(2s)$ and (5.10) yields

$$2\pi - 2\omega_0 \le \operatorname{Arg} \chi(-2s) \le 2\pi - 2\omega_0 + \delta.$$

Since

$$2\pi - 2\omega_0 + \delta = 2\pi - 2\left(\frac{\pi}{2} + \delta\right) + \delta = \pi - \delta < \pi,$$

we have $\operatorname{Arg} \chi(-2s) < \pi$, and so $\chi(-2s)$ is a member of $\chi(G) \cap \mathbb{H}$. On the other hand, since $\omega_0 \le 2\pi/3 - \epsilon < 2\pi/3$ implies that $2\pi - 2\omega_0 > \omega_0$, we have $\operatorname{Arg} \chi(-2s) > \omega_0$. However, the latter is impossible since no element of $\chi(G) \cap \mathbb{H}$ has the principal argument greater than ω_0.

Since either case leads to a contradiction, the theorem is proved. ■

For a unital algebra $\mathfrak{A}$, we denote by $\mathfrak{A}^\times$ the group of multiplicatively invertible elements of $\mathfrak{A}$. The neutral element of $\mathfrak{A}^\times$ is obviously identical with $e_{\mathfrak{A}}$.

Let G be an Abelian group, and let $\mathfrak{A}$ be a unital algebra. An $\mathfrak{A}$-valued family $\{\mathscr{G}(s)\}_{s \in G}$ is a *group* in $\mathfrak{A}$ if $\{\mathscr{G}(s)\}_{s \in G}$ is a semigroup in $\mathfrak{A}$ with the property that $\mathscr{G}(s) \in \mathfrak{A}^\times$ for each $s \in G$. An $\mathscr{L}(X)$-valued group, where X is a non-zero normed space, is called a group on X.

The importance, for our purposes, of the concept of β-number lies in the theorem that we present next. That theorem will serve as a replacement for Proposition 2.9 in the context of homomorphisms from group algebras of locally compact Abelian groups.

THEOREM 5.6. *Let G be a non-zero Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. If $\{\mathscr{G}(s)\}_{s\in G}$ is a group in $\mathfrak{A}$ such that* $\sup_{s\in G}\|\mathscr{G}(s)-e_{\mathfrak{A}}\|<\beta(G_{\mathrm{d}})$, *then $\mathscr{G}(s)=e_{\mathfrak{A}}$ for every $s\in G$.*

Before embarking on the proof, we present the necessary background material.

For an element x of a complex normed algebra $\mathfrak{A}$, we denote the spectrum of x by $\sigma(x)$; when more specificity is required, we will use the notation $\sigma_{\mathfrak{A}}(x)$ instead.

We recall that an invertible element x of a unital normed algebra is termed *doubly power bounded* if $\sup_{n\in\mathbb{Z}}\|x^n\|<\infty$. From the spectral radius formula it follows that the spectrum of a doubly power bounded element of a complex, unital Banach algebra is contained in $\mathbb{T}$.

PROPOSITION 5.7 (Gelfand's theorem). *Let $\mathfrak{A}$ be a complex, unital Banach algebra, and let x be a doubly power bounded element of $\mathfrak{A}$. If $\sigma(x)=\{1\}$, then $x=e_{\mathfrak{A}}$.*

Gelfand's theorem can be proved in a number of different ways (see, e.g., [5], [6, Theorem 1.1], [34], [65, Corollary 4.2]). The result has various generalisations, of which one is due to Hille [40] (see also [41, Theorem 4.10.1]) and is usually referred to as the Gelfand–Hille theorem; it states that, if x is an element of a complex, unital Banach algebra $\mathfrak{A}$ such that $\sigma(x)=\{1\}$, then $(x-e_{\mathfrak{A}})^r=0$ for some $r\in\mathbb{N}$ if and only if $\|x^n\|=O(n^{r-1})$, or $\|x^n\|=o(n^r)$, as $|n|\to\infty$. For an informative account of various developments related to the Gelfand–Hille theorem, see [100]; and for modern generalisations of this theorem, see [27] and the references therein.

Proof of Theorem 5.6. Let $M:=\sup_{s\in G}\|\mathscr{G}(s)\|$. Then, clearly,

$$M\le 1+\sup_{s\in G}\|\mathscr{G}(s)-e_{\mathfrak{A}}\|<1+\beta(G_{\mathrm{d}}),$$

so M is finite, and, in particular, $\mathscr{G}(s)$ is doubly power bounded for every $s\in G$. Let $\mathfrak{B}$ be the closed algebra generated by $\{\mathscr{G}(s)\mid s\in G\}$ in the Banach algebra completion of $\mathfrak{A}$. The algebra $\mathfrak{B}$ is commutative and unital, and its identity can be identified with the identity of $\mathfrak{A}$. By Proposition 5.7, to conclude the proof, we need only show that, for every $s\in G$, $\mathscr{G}(s)$ as an element of $\mathfrak{B}$ has spectrum equal to $\{1\}$.

We shall exploit the well-known characterisation of the spectrum of an element of a unital, commutative Banach algebra as the range of the element's Gelfand transform: if $\mathfrak{C}$ is a complex, unital, commutative Banach algebra and if $x\in\mathfrak{C}$, then

$$\sigma_{\mathfrak{C}}(x)=\{\phi(x)\mid\phi\in\Delta(\mathfrak{C})\}$$

(see, e.g., [22, Theorem 3.2.2]).

If $\phi\in\Delta(\mathfrak{B})$, then

$$\phi(\mathscr{G}(s+t))=\phi(\mathscr{G}(s)\mathscr{G}(t))=\phi(\mathscr{G}(s))\phi(\mathscr{G}(t))$$

for all $s,t\in G$ and

$$\phi(\mathscr{G}(0))=\phi(e_{\mathfrak{A}})=1.$$

Moreover,

$$\sup_{s\in G} |\phi(\mathscr{G}(s))| \le \|\phi\| M = M,$$

where we used the fact that $\|\phi\| = 1$ (see, e.g., [45, Lemma 2.1.5]). Since $(\phi(\mathscr{G}(s)))^n = \phi(\mathscr{G}(ns))$ for any $s\in G$ and any $n\in\mathbb{N}$, it follows that, given $n\in\mathbb{N}$, $(\sup_{s\in G}|\phi(\mathscr{G}(s))|)^n \le M$, and hence $\sup_{s\in G}|\phi(\mathscr{G}(s))| \le M^{1/n}$. Letting $n\to\infty$, we obtain

$$\sup_{s\in G} |\phi(\mathscr{G}(s))| \le 1. \tag{5.11}$$

If $s\in G$, then

$$\phi(\mathscr{G}(s))\phi(\mathscr{G}(-s)) = \phi(\mathscr{G}(s)\mathscr{G}(-s)) = \phi(\mathscr{G}(0)) = 1,$$

implying that

$$1 \le |\phi(\mathscr{G}(s))|\,|\phi(\mathscr{G}(-s))|.$$

The latter together with (5.11) yields

$$|\phi(\mathscr{G}(s))| = |\phi(\mathscr{G}(-s))| = 1.$$

We conclude that the mapping $s\mapsto \phi(\mathscr{G}(s))$ is a character on G. Now, choose any $0<\epsilon<\beta(G_{\mathrm{d}})$ so that $\sup_{s\in G}\|\mathscr{G}(s)-e_{\mathfrak{A}}\| \le \beta(G_{\mathrm{d}})-\epsilon$. Then

$$|\phi(\mathscr{G}(s))-1| = |\phi(\mathscr{G}(s)-e_{\mathfrak{A}})| \le \|\phi\|\,\|\mathscr{G}(s)-e_{\mathfrak{A}}\| \le \beta(G_{\mathrm{d}})-\epsilon$$

for all $s\in G$. Combining this with the definition of $\beta(G_{\mathrm{d}})$, we see that

$$\phi(\mathscr{G}(s)) = 1 \tag{5.12}$$

for all $s\in G$.

In view of the last relation, we obtain

$$\sigma_{\mathfrak{B}}(\mathscr{G}(s)) = \{\phi(\mathscr{G}(s)) \mid \phi\in\Delta(\mathfrak{B})\} = \{1\}$$

for all $s\in G$, which is all that we needed to complete the proof. ■

Remark 5.8. For a finite group G, the use of Proposition 5.7 in the last paragraph of the above proof can be replaced by a simpler argument. With $s\in G$ fixed arbitrarily, we have, in view of (5.12),

$$\phi(e_{\mathfrak{A}}+\mathscr{G}(s)+\cdots+(\mathscr{G}(s))^{|G|-1}) = \phi(e_{\mathfrak{A}})+\phi(\mathscr{G}(s))+\cdots+(\phi(\mathscr{G}(s)))^{|G|-1} = |G|$$

for all $\phi\in\Delta(\mathfrak{B})$. Consequently,

$$\sigma_{\mathfrak{B}}(e_{\mathfrak{A}}+\mathscr{G}(s)+\cdots+\mathscr{G}(s)^{|G|-1}) = \{|G|\},$$

and, in particular, $e+\mathscr{G}(s)+\cdots+(\mathscr{G}(s))^{|G|-1}$ is invertible in $\mathfrak{B}$. Since

$$\begin{aligned}(e_{\mathfrak{A}}-\mathscr{G}(s))(e_{\mathfrak{A}}+\mathscr{G}(s)+\cdots+(\mathscr{G}(s))^{|G|-1}) &= e_{\mathfrak{A}}-\mathscr{G}(s)^{|G|} = e_{\mathfrak{A}}-\mathscr{G}(|G|s)\\ &= e_{\mathfrak{A}}-\mathscr{G}(0) = 0,\end{aligned}$$

it follows that $\mathscr{G}(s) = e_{\mathfrak{A}}$.

5.3. Calculation of β-numbers. The main goal of this section is to calculate the β-number of an arbitrary non-zero, locally compact Abelian group.

We recall that the order of a character χ on an Abelian group G is the smallest positive integer n such that $(\chi(s))^n = 1$ for all $s \in G$. If no such n exists, then χ is said to have infinite order.

LEMMA 5.9. *Let G be a locally compact Abelian group. If $\chi \in \widehat{G}$ has infinite order, then $\chi(G)$ is dense in $\mathbb{T}$.*

Proof. The set $\chi(G)$ is a subgroup of $\mathbb{T}$, and the closure $\overline{\chi(G)}$ of $\chi(G)$ in $\mathbb{T}$ is a closed subgroup of $\mathbb{T}$. Consequently, $\overline{\chi(G)}$ is either finite or all of $\mathbb{T}$ (see, e.g., [61, Section 2, Corollary 3]).

Assume that $\overline{\chi(G)}$ is finite. Then *a fortiori* $\chi(G)$ is finite and there exists $n \in \mathbb{N}$ such that $(\chi(s))^n = 1$ for every $s \in G$. But this means that χ is of finite order, contrary to assumption. Therefore $\overline{\chi(G)}$ is not finite, and so $\overline{\chi(G)}$ is all of $\mathbb{T}$. ∎

LEMMA 5.10. *Let G be a locally compact Abelian group. If $\chi \in \widehat{G}$ has infinite order, then*

$$\sup_{s\in G} |\chi(s) - 1| = 2.$$

Proof. By Lemma 5.9, $\chi(G)$ is dense in $\mathbb{T}$. In particular, there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ in G such that $\lim_{n\in\mathbb{N}} \chi(s_n) = -1$. Clearly, $\lim_{n\in\mathbb{N}} |\chi(s_n) - 1| = 2$. Since $|\chi(s)-1| \le 2$ for each $s \in G$, it follows that $\sup_{n\in\mathbb{N}} |\chi(s_n)-1| = 2$, and further that $\sup_{s\in G} |\chi(s)-1| = 2$. ∎

LEMMA 5.11. *Let G be a non-zero, locally compact Abelian group. If $\chi \in \widehat{G}\setminus\{1_G\}$ is such that $\chi^m = 1_G$ for some $m \in \mathbb{N}$, then, letting $q := |\chi(G)|$, we have the following:*

(i) $q > 1$;
(ii) q *divides* m;
(iii) $\displaystyle\sup_{s\in G} |\chi(s) - 1| = 2\sin\left(\frac{\pi[q/2]}{q}\right)$.

Proof. As $\chi \neq 1_G$, we have $m > 1$ and $q > 1$. Given that $(\chi(s))^m = 1$ for each $s \in G$, it is clear that $\chi(G) \subset \mathbb{U}_m$, where $\mathbb{U}_m$ is the multiplicative group of mth roots of unity. Moreover, $\chi(G)$ is a subgroup of $\mathbb{U}_m$. Since $\mathbb{U}_m$ is cyclic (isomorphic to $\mathbb{Z}_m$) and since a subgroup of a cyclic group is cyclic, it follows that $\chi(G)$ is cyclic. Let r be the smallest integer with $1 \le r \le m$ such that $\exp(2\pi i r/m)$ is a generator of $\chi(G)$. Then, clearly, $qr = m$. We see that q divides m and, moreover, that $\exp(2\pi i r/m) = \exp(2\pi i/q)$. The latter implies that

$$\chi(G) = \left\{\exp\left(\frac{2\pi i l}{q}\right) \,\middle|\, l = 0, \ldots, q-1\right\}.$$

Consequently,

$$\{|\chi(s) - 1| \mid s \in G\} = \left\{\left|\exp\left(\frac{2\pi i l}{q}\right) - 1\right| \,\middle|\, l = 0, \ldots, q-1\right\},$$

and since the largest of the numbers

$$\left|\exp\left(\frac{2\pi i l}{q}\right) - 1\right| \quad (l = 0, \ldots, q-1)$$

is $2\sin(\pi[q/2]/q)$, the lemma follows. ∎

Corollary 5.12. *Let G be a non-zero, locally compact Abelian group. If $\chi \in \widehat{G} \setminus \{1_G\}$ has order p, where p is a prime number, then*

$$\sup_{s\in G} |\chi(s) - 1| = 2\sin\left(\frac{\pi[p/2]}{p}\right).$$

Proof. The corollary is an immediate consequence of Lemma 5.11 and the fact that the only divisor of p greater than 1 is p itself. ■

Lemma 5.13. *Let G be a locally compact Abelian group. If $\widehat{G}$ contains an element whose order is finite and has a prime factor p, then $\widehat{G}$ contains an element of order p.*

Proof. According to a classical theorem of Cauchy, if H is a finite group (Abelian or not) and if p is a prime factor of $|H|$, then H contains an element of order p (cf. [43, Theorem 5.2]). Suppose that $\chi \in \widehat{G}$ has finite order m and that p is a prime factor of m. The lemma follows at once by applying Cauchy's theorem to the group $\{\chi^k \mid k = 0, \dots, m-1\}$. ■

We are almost ready to present the main result of this section. However, before proceeding, we still need to lay out a few more prerequisites.

Let G be a non-zero, locally compact Abelian group. Then the following mutually exclusive cases may occur:

(i) each element of $\widehat{G} \setminus \{1_G\}$ is either of infinite order or of order 2^k for some $k \in \mathbb{N}$;
(ii) $\widehat{G} \setminus \{1_G\}$ contains an element whose order is finite and has an odd prime factor.

Let $\mathscr{P}_{\widehat{G}}$ be the set of odd prime numbers p such that there exists a member of $\widehat{G}$ having order p. We note that, if case (ii) holds, then, by Lemma 5.13, $\mathscr{P}_{\widehat{G}}$ is non-empty, and, in particular, one can speak about the smallest element of $\mathscr{P}_{\widehat{G}}$.

We are now ready to state the aforementioned main result.

Theorem 5.14. *Let G be a non-zero, locally compact Abelian group.*

(i) *If each element of $\widehat{G} \setminus \{1_G\}$ is either of infinite order or of order 2^k for some $k \in \mathbb{N}$, then $\beta(G) = 2$.*
(ii) *If $\widehat{G} \setminus \{1_G\}$ contains an element whose order is finite and has an odd prime factor, then*

$$\beta(G) = 2\sin\left(\frac{\pi[r/2]}{r}\right),$$

where r is the smallest element of $\mathscr{P}_{\widehat{G}}$.

Proof. (i) If $\chi \in \widehat{G}$ has infinite order, then, by Lemma 5.9, $\chi(G)$ is dense in $\mathbb{T}$ and

$$\sup_{s\in G} |\chi(s) - 1| = 2.$$

If $\chi \in \widehat{G}$ has order 2^k, then, by Lemma 5.11, there exists $q = 2^j$ with $1 \le j \le k$ such that

$$\sup_{s\in G} |\chi(s) - 1| = 2\sin\left(\frac{\pi[q/2]}{q}\right) = 2\sin\left(\frac{\pi}{2}\right) = 2.$$

In both cases

$$\sup_{s\in G} |\chi(s) - 1| = 2,$$

and this immediately yields the assertion in question.

(ii) Suppose now that there exists a member of $\widehat{G}\setminus\{1_G\}$ whose order is finite and has an odd prime factor. With r being the smallest element of $\mathscr{P}_{\widehat{G}}$, let $\zeta\in\widehat{G}$ have order r. By Corollary 5.12,

$$\sup_{s\in G}|\zeta(s)-1| = 2\sin\left(\frac{\pi[r/2]}{r}\right).$$

We want to prove that

$$\sup_{s\in G}|\zeta(s)-1| = \beta(G).$$

By definition of $\beta(G)$, the left-hand side here does not exceed the right-hand side, and therefore it suffices to show that, if $\gamma\in\widehat{G}$, then

$$\sup_{s\in G}|\zeta(s)-1| \le \sup_{s\in G}|\gamma(s)-1|.$$

Since $2\sin(\pi[r/2]/r)\le 2$, the latter inequality is clear when the right-hand side equals 2. By the argument used to prove statement (i), we conclude that this is the case when γ is of order 2^k for some integer k or when γ is of infinite order.

Thus, we are left with the case where the order of γ is (finite and) odd. By Lemma 5.11, then there is a necessarily odd positive integer q such that

$$\sup_{s\in G}|\gamma(s)-1| = 2\sin\left(\frac{\pi[q/2]}{q}\right).$$

If p is a prime factor of q, then, clearly, p is odd, and, by Lemma 5.13, p belongs to $\mathscr{P}_{\widehat{G}}$. This implies that $r\le p\le q$. As the sequence $\{\sin(\pi n/(2n+1))\}_{n\in\mathbb{N}}$ is increasing, the function $n\mapsto\sin(\pi[n/2]/n)$ is increasing on the set of odd positive integers. It follows that

$$2\sin\left(\frac{\pi[q/2]}{q}\right)\ge 2\sin\left(\frac{\pi[r/2]}{r}\right),$$

and we are done. ∎

THEOREM 5.15. *If G is a non-zero, divisible Abelian group, then $\beta(G_{\mathrm{d}})=2$.*

Proof. In view of Theorem 5.14, it suffices to show that every $\chi\in\widehat{G}\setminus\{1_G\}$ has infinite order. Assume to the contrary that some $\chi\in\widehat{G}\setminus\{1_G\}$ has finite order k. Then $\chi(kt)=(\chi(t))^k=1$ for all $t\in G$. Since, by the divisibility of G, each s in G can be represented as kt for some $t\in G$, it follows that $\chi(s)=1$ for all $s\in G$, contrary to the assumption that χ is non-trivial. ∎

THEOREM 5.16. *We have $\beta(\mathbb{Z})=\sqrt{3}$.*

Proof. As is well known, $\widehat{\mathbb{Z}}$ can be identified with the circle group $\mathbb{T}$. Among the elements of $\mathbb{T}\setminus\{1\}$, some are of finite order and others are of infinite order. The orders of finite-order elements of $\mathbb{T}\setminus\{1\}$ range over all of the positive integers greater than 1. Among these, the smallest odd prime order is 3. Hence, in view of Theorem 5.14,

$$\beta(\mathbb{Z}) = 2\sin\left(\frac{\pi[3/2]}{3}\right)=\sqrt{3}. \ ∎$$

REMARK. With the theorem just proved, we can now derive anew the lower bound given in Theorem 5.5. To this end, it suffices to show that, if G is a non-zero Abelian group, then $\beta(G_{\mathrm{d}}) \geq \sqrt{3}$. Note that, if $\chi \in \widehat{G_{\mathrm{d}}} \setminus \{1_G\}$ and if $s \in G$ is such that $\chi(s) \neq 1$, then

$$\|\chi - 1_G\|_\infty \geq \sup_{n\in\mathbb{Z}} |\chi(ns) - 1| = \sup_{n\in\mathbb{Z}} |\chi(s)^n - 1| \geq \beta(\mathbb{Z}).$$

Hence $\beta(G_{\mathrm{d}}) \geq \beta(\mathbb{Z})$, and now the bound in question, $\beta(G_{\mathrm{d}}) \geq \sqrt{3}$, is an immediate consequence of Theorem 5.16.

Given two groups G_1 and G_2, we use the notation $G_1 \cong G_2$ to indicate that G_1 and G_2 are isomorphic.

THEOREM 5.17. *Let G be a non-zero, finite Abelian group isomorphic to $\mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{e_t}}$, where $p_1, \ldots, p_t$ are prime numbers and $e_1, \ldots, e_t$ are positive integers. Let $r = \min(\{p_1, \ldots, p_t\} \setminus \{2\})$ in the case where some of the p_i's are greater than 2. Then*

$$\beta(G) = \begin{cases} 2 & \text{whenever } p_1 = \cdots = p_t = 2, \\ 2\sin\left(\dfrac{\pi[r/2]}{r}\right) & \text{otherwise.} \end{cases}$$

Proof. Since G is finite, we have $\widehat{G} \cong G$ (the isomorphism is not a canonical one; see, e.g., [32, Corollary 4.8]), and so $\widehat{G} \cong \mathbb{Z}_{p_1^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p_t^{e_t}}$. If $p_1 = \cdots = p_t = 2$, then $|\widehat{G}|$ is a power of 2, and consequently, the order of every non-trivial element of $\widehat{G}$ is a power of 2. Hence, by Theorem 5.14, $\beta(G) = 2$.

Suppose now that some of the p_i's are odd prime numbers. Since $|\widehat{G}| = p_1^{e_1} \ldots p_t^{e_t}$, it is clear that, if an element of $\widehat{G}$ has order which is a prime, then the order of this element is one of the numbers $p_1, \ldots, p_t$. By Cauchy's theorem, for each $i = 1, \ldots, t$, there exists an element of $\widehat{G}$ of order p_i. Let r be the smallest of the odd primes among the p_i's. Then, by Theorem 5.14,

$$\beta(G) = 2\sin\left(\frac{\pi[r/2]}{r}\right). \blacksquare$$

COROLLARY 5.18. *If p is a prime number, then*

$$\beta(\mathbb{Z}_p) = 2\sin\left(\frac{\pi[p/2]}{p}\right).$$

REMARK. Bearing in mind that every non-zero, finite Abelian group G is isomorphic to a direct sum of the form $\mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_t}$, where $q_1, \ldots, q_t$ are (not necessarily distinct) powers of prime numbers, we see that Theorem 5.17 gives the formula for $\beta(G)$ for every non-zero, finite Abelian group G.

5.4. Relationship with homomorphisms. In this section, we exhibit a relationship between β-numbers and homomorphisms from group algebras.

Let G be a locally compact Abelian group with dual group $\widehat{G}$. Given $f \in L^1(G)$, the *Fourier transform* of f is the function $\chi \mapsto \mathscr{F}_\chi(f)$ on $\widehat{G}$ defined by

$$\mathscr{F}_\chi(f) = \int_G f(s)\overline{\chi(s)}\,\mathrm{d}s \qquad (\chi \in \widehat{G}).$$

For each $\chi \in \widehat{G}$, the mapping $\mathscr{F}_\chi\colon f \mapsto \mathscr{F}_\chi(f)$ is a character on $L^1(G)$. Conversely, every character on $L^1(G)$ coincides with $\mathscr{F}_\chi$ for some uniquely determined $\chi \in \widehat{G}$ (see, e.g., [38, Corollary 23.7]). It is plain that $\mathscr{F}_{1_G}$ is the fundamental character on $L^1(G)$; this character is usually called the *augmentation character* on $L^1(G)$ (see, e.g., [21, Definition 3.3.29]).

Given $s \in G$, we denote by T_s the operator of *translation* by s on $L^1(G)$, defined by

$$(T_s f)(t) = f(t+s) \qquad (f \in L^1(G), \text{ a.e. } t \in G). \tag{5.13}$$

It is straightforward to verify that the family $\{T_s\}_{s\in G}$ is a strongly continuous group on $L^1(G)$ and, moreover, that

$$T_s(f \star g) = T_s f \star g = f \star T_s g$$

for all $s \in G$ and all $f, g \in L^1(G)$.

We recall that the algebra $L^1(G)$ possesses an approximate identity $\{u_\iota\}_{\iota\in I}$ satisfying $\|u_\iota\|_1 = 1$ for each $\iota \in I$ (see, e.g., [38, Theorem 20.27]).

The proof of the following result proceeds along the same lines as the proof of Proposition 3.4 and is therefore omitted.

PROPOSITION 5.19. *Let G be a locally compact Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. Let $H\colon L^1(G) \to \mathfrak{A}$ be a non-zero, continuous homomorphism. Then:*

(i) *there exists a unique group $\mathscr{G} = \{\mathscr{G}(s)\}_{s\in G}$ on $\operatorname{Ran}(H)$ such that*

$$\mathscr{G}(s)H(f) = H(T_s f)$$

for each $s \in G$ and each $f \in L^1(G)$; the group $\mathscr{G}$ is strongly continuous and satisfies $\sup_{s\in G} \|\mathscr{G}(s)\| \le \|H\|$;

(ii) *if $\{u_\iota\}_{\iota\in I}$ is an approximate identity for $L^1(G)$, then*

$$\mathscr{G}(s)x = \lim_{\iota\in I} H(T_s u_\iota)x \tag{5.14}$$

for each $s \in G$ and each $x \in \operatorname{Ran}(H)$;

(iii) *with L denoting the left regular representation of $\mathfrak{A}$, the mapping $(L \circ H)\restriction_{\operatorname{Ran}(H)}$ admits the representation*

$$\big((L \circ H)\restriction_{\operatorname{Ran}(H)}\big)(f)x =: H(f)x = \int_G f(s)\mathscr{G}(s)x \,\mathrm{d}m_G(s) \tag{5.15}$$

for each $f \in L^1(G)$ and each $x \in \operatorname{Ran}(H)$.

Our main result in this section is as follows.

THEOREM 5.20. *Let G be a non-zero, locally compact Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. If $\chi \in \widehat{G}$ and if $H\colon L^1(G) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| < \beta(G_{\mathrm{d}})$, then there exists an idempotent e in $\mathfrak{A}$ such that $H = e \otimes \mathscr{F}_\chi$.*

Proof. The proof follows closely that of Theorem 3.5 and is similarly structured.

STEP 1. If $H = 0$, then $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| < \beta(G_{\mathrm{d}})$, since $\|\mathscr{F}_\chi\| = \|\chi\|_\infty = 1$ implies that $\|e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| = \|e_{\mathfrak{A}}\|\,\|\mathscr{F}_\chi\| = 1$ and since $\beta(G_{\mathrm{d}}) \ge \sqrt{3}$ by Theorem 5.5. Also, if $H = 0$, then $H = 0 \otimes \mathscr{F}_\chi$, where the last 0 denotes the zero element of $\mathfrak{A}$. We see that the theorem holds when H is zero. From now on we may therefore suppose that H is non-zero. Choose a contractive approximate identity $\{u_\iota\}_{\iota\in I}$ for $L^1(G)$. Let $\{\mathscr{G}(s)\}_{s\in G}$ be the group on $\operatorname{Ran}(H)$ whose existence is established in Proposition 5.19. If $s \in G$, if $x \in \operatorname{Ran}(H)$, and if $\iota \in I$, then

$$\begin{aligned}\|H(T_s u_\iota)x - (e_{\mathfrak{A}} \otimes \mathscr{F}_\chi)(T_s u_\iota)x\| &\le \|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| \, \|T_s u_\iota\|_1 \|x\| \\ &\le \|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| \, \|x\|. \end{aligned} \tag{5.16}$$

Since χ is continuous, we have $\lim_{\iota \in I} \mathscr{F}_\chi(T_s u_\iota) = \overline{\chi(s)}$, and so

$$\lim_{\iota \in I} (e_{\mathfrak{A}} \otimes \mathscr{F}_\chi)(T_s u_\iota)x = \overline{\chi(s)}x.$$

Combining this with (5.14), we deduce from (5.16) that

$$\|\mathscr{G}(s)x - \overline{\chi(s)}x\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| \, \|x\|.$$

Consequently,

$$\sup_{s \in G} \|\mathscr{G}(s) - \overline{\chi(s)} I_{\mathrm{Ran}(H)}\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\|,$$

and further, since χ is unitary,

$$\sup_{s \in G} \|\chi(s)\mathscr{G}(s) - I_{\mathrm{Ran}(H)}\| \le \|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\|.$$

This together with the assumption that $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| < \beta(G_{\mathrm{d}})$ yields

$$\sup_{s \in G} \|\chi(s)\mathscr{G}(s) - I_{\mathrm{Ran}(H)}\| < \beta(G_{\mathrm{d}}).$$

An application of Theorem 5.6 now implies that

$$\chi(s)\mathscr{G}(s) = I_{\mathrm{Ran}(H)}$$

for all $s \in G$. Hence, immediately, $\mathscr{G}(s) = \overline{\chi(s)} I_{\mathrm{Ran}(H)}$ for all $s \in G$, and further, by (5.15),

$$H(f)x = \mathscr{F}_\chi(f)x \tag{5.17}$$

for all $f \in L^1(G)$ and all $x \in \mathrm{Ran}(H)$.

Step 2. If $f \in L^1(G)$, then

$$H(f) = \lim_{\iota \in I} H(u_\iota)H(f) = \lim_{\iota \in I} H(f)H(u_\iota). \tag{5.18}$$

For each $\iota \in I$, putting $x = H(u_\iota)$ in (5.17) yields

$$H(f)H(u_\iota) = \mathscr{F}_\chi(f)H(u_\iota).$$

Combining this with (5.18), we find that

$$H(f) = \lim_{\iota \in I} \mathscr{F}_\chi(f)H(u_\iota). \tag{5.19}$$

Now, arguing as in the proof of Theorem 3.5, we deduce that the limit $\lim_{\iota \in I} H(u_\iota)$ exists and is an idempotent in $\mathrm{Ran}(H)$. This together with (5.19) establishes the theorem. ∎

Remark 5.21. Comparing Theorem 5.20 with Theorem 3.5, it is natural to conjecture that a stronger version of Theorem 5.20 holds, namely one to the effect that, if $\chi \in \widehat{G}$ and if $H \colon L^1(G) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| < \beta(G_{\mathrm{d}})$, then $H = e_{\mathfrak{A}} \otimes \mathscr{F}_\chi$. This conjecture is, however, false.

To see this, consider a unital normed algebra $\mathfrak{A}_0$, and let $\mathfrak{A}_0 \oplus \mathfrak{A}_0$ be the direct sum of two copies of $\mathfrak{A}_0$, endowed with the norm

$$\|(x, y)\| = \max(\|x\|, \|y\|) \quad (x, y \in \mathfrak{A}_0).$$

Let $\mathfrak{A} = \mathscr{L}(\mathfrak{A}_0 \oplus \mathfrak{A}_0)$. Then $\mathfrak{A}$ is a unital normed algebra with identity $I_{\mathfrak{A}_0 \oplus \mathfrak{A}_0}$. For each $\lambda \in \mathbb{C}$, let e_λ be the element of $\mathfrak{A}$ defined by

$$e_\lambda(x, y) = (x + \lambda y, 0) \quad (x, y \in \mathfrak{A}_0).$$

In matrix form,

$$e_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

It is readily verified that e_λ is idempotent and $\|e_\lambda\| = 1 + |\lambda|$. In particular, if $\mu \geq 1$, then $e_{\mu-1}$ is an idempotent in $\mathfrak{A}$ whose norm is equal to μ.

In view of the above and the fact that, by Theorem 5.5, $\beta(G_{\mathrm{d}}) > 1$, there exists an idempotent e in $\mathfrak{A}$ such that $1 < \|e\| < \beta(G_{\mathrm{d}})$. Pick $\chi \in \widehat{G}$ and set

$$H := (e_{\mathfrak{A}} - e) \otimes \mathscr{F}_\chi.$$

Then

$$\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| = \|e \otimes \mathscr{F}_\chi\| = \|e\| \, \|\mathscr{F}_\chi\| = \|e\| < \beta(G_{\mathrm{d}}).$$

But H is clearly different from $e_{\mathfrak{A}} \otimes \mathscr{F}_\chi$. This disproves the conjecture.

5.5. Applications. Based on the material from the last and previous sections, we now draw further conclusions regarding isolability properties of homomorphisms.

Theorem 5.20 coupled with Theorem 5.5 guarantees that, if G is a non-zero, locally compact Abelian group, then $L^1(G)$ has property $(\mathrm{P}_{\mathscr{F}_\chi})$ for every $\chi \in \widehat{G}$ (to recall the definition of the properties in question, see the statement of Theorem 2.16). This, together with the fact that $\|\mathscr{F}_\chi\| = 1$ for each $\chi \in \widehat{G}$, leads, by virtue of Theorem 2.16, to the following result.

Theorem 5.22. *Let G be a non-zero, locally compact Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. If $\chi \in \widehat{G}$, then $e_{\mathfrak{A}} \otimes \mathscr{F}_\chi \colon L^1(G) \to \mathfrak{A}$ is totally isolated.*

We also immediately deduce the following theorem.

Theorem 5.23. *Let G be a non-zero, locally compact Abelian group, let $\mathfrak{A} \in \mathscr{A}_{\mathrm{wni}}$, and let $\chi \in \widehat{G}$. If $H \colon L^1(G) \to \mathfrak{A}$ is a non-zero homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_\chi\| < \beta(G_{\mathrm{d}})$, then $H = e_{\mathfrak{A}} \otimes \mathscr{F}_\chi$.*

Proof. By Theorem 5.20, there exists an idempotent e in $\mathfrak{A}$ such that $H = e \otimes \mathscr{F}_\chi$. Since $\mathfrak{A}$ has no non-trivial idempotent, e is either the zero element or the identity element of $\mathfrak{A}$. Since H is non-zero, we necessarily have $e = e_{\mathfrak{A}}$, and this implies that $H = e_{\mathfrak{A}} \otimes \mathscr{F}_\chi$. ∎

An application of Theorem 5.23 with χ taken to be 1_G establishes the following result.

Theorem 5.24. *If G is a non-zero, locally compact Abelian group, then $\beta(G_{\mathrm{d}}) \leq \alpha(L^1(G))$.*

Combining this theorem with Theorem 5.5 yields the following corollary.

Corollary 5.25. *If G is a non-zero, locally compact Abelian group, then $\alpha(L^1(G)) \geq \sqrt{3}$.*

As $\beta(G_{\mathrm{d}}) \leq \beta(G)$ for every non-zero, locally compact Abelian group G, the following result can be viewed as a kind of partial converse of Theorem 5.24.

Theorem 5.26. *If G is a non-zero, locally compact Abelian group, then $\alpha(L^1(G)) \leq \beta(G)$.*

Proof. For each $\chi \in \hat{G}$, $\mathscr{F}_\chi$ is a non-zero, complex-valued homomorphism from $L^1(G)$, and

$$\|\mathscr{F}_\chi - \mathscr{F}_{1_G}\| = \sup_{s \in G} |\chi(s) - 1|. \tag{5.20}$$

Moreover, if $\chi \neq 1_G$, then $\mathscr{F}_\chi \neq \mathscr{F}_{1_G}$. Since $\mathscr{F}_{1_G}$ is the fundamental character on $L^1(G)$ and since $\mathbb{C}$ is an algebra with no non-trivial idempotents, it follows that

$$\alpha(L^1(G)) \leq \inf_{\chi \in \widehat{G} \setminus \{1_G\}} \|\mathscr{F}_\chi - \mathscr{F}_{1_G}\|.$$

In view of (5.20), this implies that $\alpha(L^1(G)) \leq \beta(G)$. ∎

Combining Theorems 5.24 and 5.26, we obtain the following result.

Theorem 5.27. *If G is a non-zero, locally compact Abelian group such that $\beta(G_{\mathrm{d}}) = \beta(G)$, then $\alpha(L^1(G)) = \beta(G)$.*

Since there is no distinction between $\beta(G_{\mathrm{d}})$ and $\beta(G)$ when G is a non-zero, discrete Abelian group, we immediately obtain the following corollary.

Corollary 5.28. *If G is a non-zero, discrete Abelian group, then $\alpha(\ell^1(G)) = \beta(G)$.*

Remark. If G is a non-zero, finite Abelian group, then $\alpha(\ell^1(G)) = \beta(G)$ by Corollary 5.28, and $\beta(G)$ in turn can be explicitly calculated using Theorem 5.17. We thus have a general device for determining the α-number of the group algebra of an arbitrary non-zero, finite Abelian group. In Section 5.1 we calculated $\alpha(\ell^1(G))$ directly in the case where G is one of the groups $\mathbb{Z}_2$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (the Klein four-group), and $\mathbb{Z}_3$. Unsurprisingly and reassuringly, we can retrieve the same results using the general device just mentioned. Indeed, in accordance with Theorem 5.17, $\beta(\mathbb{Z}_2) = 2$, $\beta(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = 2$, and $\beta(\mathbb{Z}_3) = 2\sin(\pi[3/2]/3) = \sqrt{3}$, and further, by Corollary 5.28, $\alpha(\ell^1(\mathbb{Z}_2)) = 2$, $\alpha(\ell^1(\mathbb{Z}_2 \oplus \mathbb{Z}_2)) = 2$, and $\alpha(\ell^1(\mathbb{Z}_3)) = \sqrt{3}$, the three latter values being exactly the values obtained in Section 5.1.

The following result is an immediate consequence of Theorem 5.16 and Corollary 5.28.

Theorem 5.29. *We have $\alpha(l^1(\mathbb{Z})) = \sqrt{3}$.*

Another readily-obtained result is as follows.

Theorem 5.30. *If G is a non-zero, divisible, locally compact Abelian group, then $\alpha(L^1(G)) = 2$.*

Proof. Since $\beta(G) \leq 2$, $\beta(G_{\mathrm{d}}) \leq \beta(G)$, and, by Theorem 5.15, $\beta(G_{\mathrm{d}}) = 2$, we see that $\beta(G_{\mathrm{d}}) = \beta(G) = 2$. Now the theorem follows from Theorem 5.27. ∎

As an immediate consequence, we obtain the following theorem.

Theorem 5.31. *We have $\alpha(L^1(\mathbb{R})) = \alpha(l^1(\mathbb{R})) = 2$.*

6. Convolution algebras of integrable even functions on groups

Another, and final, class of ordered AL-algebras that we shall consider in connection with the theme of isolability properties of homomorphisms is formed by the convolution algebras of Haar integrable, even functions on locally compact Abelian groups.

Let G be a locally compact Abelian group. Let $L^1_{\mathrm{e}}(G)$ be the space of all even elements of $L^1(G)$:

$$L^1_{\mathrm{e}}(G) = \{f \in L^1(G) \mid f(s) = f(-s) \text{ for a.e. } s \in G\}.$$

The space $L^1_{\mathrm{e}}(G)$ is a closed subalgebra and a sublattice of $L^1(G)$. One verifies at once that $L^1_{\mathrm{e}}(G)$ is a complex ordered AL-algebra. It is algebras of type $L^1_{\mathrm{e}}(G)$ and their homomorphisms that will be of interest for us in this chapter.

6.1. Background. Let G be an Abelian group, and let $\mathfrak{A}$ be a unital algebra. An $\mathfrak{A}$-valued family $\{\mathscr{C}(s)\}_{s\in G}$ is a *cosine function*, or *cosine family*, in $\mathfrak{A}$ if:

(i) $2\mathscr{C}(s)\mathscr{C}(t) = \mathscr{C}(s+t) + \mathscr{C}(s-t)$ for all $s, t \in G$;
(ii) $\mathscr{C}(0) = e_{\mathfrak{A}}$.

An $\mathscr{L}(X)$-valued cosine function, where X is a non-zero normed space, is called a cosine function on X. It is straightforward to see that, if $\{\mathscr{C}(s)\}_{s\in G}$ is a cosine function in $\mathfrak{A}$, then (i) every pair of elements of the range of the function commutes: $\mathscr{C}(s)\mathscr{C}(t) = \mathscr{C}(t)\mathscr{C}(s)$ for all $s, t \in G$; (ii) the function is even: $\mathscr{C}(s) = \mathscr{C}(-s)$ for all $s \in G$.

Let G be an Abelian group. By a *quasi-character* on G we mean a homomorphism from G into the multiplicative group $\mathbb{C}^{\times}$ of complex numbers (see, e.g., [52, p. 278] for this terminology).

Our subsequent development will critically involve scalar-valued cosine functions on Abelian groups. The following proposition lists all relevant, for our purposes, properties of such functions:

PROPOSITION 6.1 (Kannappan [46]). *Let G be an Abelian group, and let $c\colon G \to \mathbb{C}$.*

(i) *c is a cosine function if and only if there exists a quasi-character χ on G such that*
$$c = \tfrac{1}{2}(\chi + \check{\chi}), \tag{6.1}$$
where $\check{\chi}$ is the quasi-character on G defined by $\check{\chi}(s) = \chi(-s)$ for all $s \in G$.
(ii) *Suppose that c is of the form given in* (6.1), *where χ is a quasi-character on G. Then:*
   (a) *if* (6.1) *also holds with χ being replaced by another quasi-character γ on G, then either $\gamma = \chi$ or $\gamma = \check{\chi}$;*
   (b) *c is bounded if and only if χ is bounded, and this happens precisely when χ is unitary* (*in which case χ is a character on G*)*;*
   (c) *if G is a topological group, then c is continuous* (*at a single point, or, equivalently, everywhere*) *if and only if χ is continuous.*

Proof. Assertion (i) and parts (a) and (c) of assertion (ii) follow from Theorems 2, 3, and 1 of [46], respectively. For part (b) of assertion (ii), see, e.g., the proof of [13, Theorem 10]. ∎

6.2. The γ-numbers. We now introduce certain numerical characteristics that will serve as counterparts of β-numbers in the context of algebras of the form $L^1_{\mathrm{e}}(G)$.

Let G be a locally compact Abelian group. We denote by $\mathrm{Cos}(G)$ the set of all complex-valued, continuous, bounded cosine functions on G. In view of (i) and (ii)(b) of Proposition 6.1, each member of $\mathrm{Cos}(G)$ is a real function with values between -1 and 1. In the subsequent analysis, the set $\mathrm{Cos}(G)$ will play a role analogous to that played by the character group $\widehat{G}$ for homomorphisms from $L^1(G)$. One distinguished member of $\mathrm{Cos}(G)$ is 1_G, the trivial cosine function on G. When G is non-zero, $\mathrm{Cos}(G)$ contains at least one more extra element. Indeed, in that case $\widehat{G} \setminus \{1_G\}$ is non-empty; and if $\chi \in \widehat{G} \setminus \{1_G\}$, then $(\chi + \overline{\chi})/2$ $(= (\chi + \check{\chi})/2)$ is in $\mathrm{Cos}(G)$ and is different from 1_G.

If G is a non-zero, locally compact Abelian group and if $c \in \mathrm{Cos}(G)$, we let

$$\gamma(G, c) := \inf_{\substack{\tilde{c} \in \mathrm{Cos}(G) \\ \tilde{c} \neq c}} \sup_{s \in G} |c(s) - \tilde{c}(s)|.$$

We shall refer to $\gamma(G, c)$ as the *γ-number* of the pair (G, c). Note that $\gamma(G_{\mathrm{d}}, c) \leq \gamma(G, c)$ whenever $c \in \mathrm{Cos}(G)$, which immediately results from the relation $\mathrm{Cos}(G) \subset \mathrm{Cos}(G_{\mathrm{d}})$.

The result stated next reveals the significance of γ-numbers for isolability considerations. It will be used as a counterpart of Theorem 5.6 in the current context.

THEOREM 6.2. *Let G be a non-zero Abelian group, let $\mathfrak{A}$ be a unital normed algebra, and let $c \in \mathrm{Cos}(G_{\mathrm{d}})$. If $\{\mathscr{C}(s)\}_{s \in G}$ is a cosine function in $\mathfrak{A}$ such that $\sup_{s \in G} \|\mathscr{C}(s) - c(s) e_{\mathfrak{A}}\| < \gamma(G_{\mathrm{d}}, c)$, then $\mathscr{C}(s) = c(s) e_{\mathfrak{A}}$ for all $s \in G$.*

The following generalisation of Gelfand's theorem will be instrumental in the proof of the above theorem.

PROPOSITION 6.3 ([14, Theorem 6]). *Let $\mathfrak{A}$ be a complex, unital Banach algebra, and let a be a doubly power bounded element of $\mathfrak{A}$ with a finite spectrum $\sigma(a) = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{T}$, $\lambda_k \neq \lambda_l$ for $k \neq l$. Then there exist idempotents $e_1, \ldots, e_n$ in $\mathfrak{A}$ such that $\sum_{k=1}^n e_k = e_{\mathfrak{A}}$, $e_k e_l = 0$ for $k \neq l$, and $a = \sum_{k=1}^n \lambda_k e_k$.*

Proof of Theorem 6.2. We adapt the argument of [14, Theorem 7] and proceed in three steps.

STEP 1. Let $\ell^\infty(G, \mathfrak{A})$ be the space of all bounded functions from G to $\mathfrak{A}$, endowed with the norm

$$\|x\|_\infty = \sup_{s \in G} \|x(s)\| \quad (x \in \ell^\infty(G, \mathfrak{A})).$$

For $x \in \mathfrak{A}$, let $\underline{x}$ denote the constant function on G with value x. For each $s \in G$, we define a linear operator $C(s)$ on $\ell^\infty(G, \mathfrak{A})$ by

$$(C(s)x)(t) = \mathscr{C}(s)[x(t)] \quad (x \in \ell^\infty(G, \mathfrak{A}),\, t \in G).$$

Clearly, $C(s)$ is bounded, with $\|C(s)\| \leq \|\mathscr{C}(s)\|$. Since $C(s)\underline{e_{\mathfrak{A}}} = \underline{\mathscr{C}(s)}$ and $\|\underline{e_{\mathfrak{A}}}\|_\infty = 1$, we see that in fact $\|C(s)\| = \|\mathscr{C}(s)\|$. It is plain that $C = \{C(t)\}_{s \in G}$ is a cosine family on $\ell^\infty(G, \mathfrak{A})$. For each $s \in G$, we have

$$\big((C(s) - c(s) I_{\ell^\infty(G, \mathfrak{A})})x\big)(t) = (\mathscr{C}(s) - c(s) e_{\mathfrak{A}})[x(t)]$$

for all $x \in \ell^\infty(G, \mathfrak{A})$ and all $t \in G$, and this implies, as above, that

$$\|C(s) - c(s) I_{\ell^\infty(G, \mathfrak{A})}\| = \|\mathscr{C}(s) - c(s) e_{\mathfrak{A}}\|.$$

Choose any $0 < \epsilon < \gamma(G_{\mathrm{d}}, c)$ so that $\|\mathscr{C}(s) - c(t) e_{\mathfrak{A}}\| \leq \gamma(G_{\mathrm{d}}, c) - \epsilon$ for all $s \in G$. Then, clearly,

$$\|C(s) - c(s) I_{\ell^\infty(G, \mathfrak{A})}\| \leq \gamma(G_{\mathrm{d}}, c) - \epsilon \tag{6.2}$$

for all $s \in G$.

STEP 2. We next show that, when restricted to a certain subspace of $\ell^\infty(G,\mathfrak{A})$, C admits a representation in terms of some group of isometries acting on that subspace.

Reusing our earlier notation (cf. (5.13)), given $s \in G$, we denote by T_s the operator of translation by s on $\ell^\infty(G,\mathfrak{A})$, defined by

$$(T_s x)(t) = x(t+s) \quad (x \in \ell^\infty(G,\mathfrak{A}),\ t \in G).$$

Clearly, T_s is a surjective linear isometry, with inverse T_{-s}. Note that, since

$$\|\mathscr{C}(s)\| \le \|\mathscr{C}(s) - c(s)e_{\mathfrak{A}}\| + |c(s)|\,\|e_{\mathfrak{A}}\| \le \gamma(G_{\mathrm{d}}, c) - \epsilon + 1$$

for all $s \in G$, $\mathscr{C} = \{\mathscr{C}(s)\}_{s\in G}$ is bounded, or equivalently, $\mathscr{C}$ is a member of $\ell^\infty(G,\mathfrak{A})$. Let Z be the linear space of all functions z in $\ell^\infty(G,\mathfrak{A})$ of the form

$$z = \sum_{k=1}^{n} \alpha_k T_{s_k}\mathscr{C},$$

where $\alpha_k \in \mathbb{C}$ and $s_k \in G$ for $k = 1,\dots,n$. It is clear that Z is invariant under T_s for every $s \in G$. For each $s \in G$, let

$$\tilde{T}_s = T_s\restriction_Z.$$

The family $\tilde{T} = \{\tilde{T}_s\}_{s\in G}$ is a group in the normed algebra $\mathscr{L}(Z)$. Moreover, since $\|\tilde{T}_s\| \le 1$, $\|\tilde{T}_{-s}\| \le 1$, and $1 = \|I_Z\| \le \|\tilde{T}_s\|\,\|\tilde{T}_{-s}\|$ for every $s \in G$, we have $\|\tilde{T}_s\| = 1$ for every $s \in G$.

For each $s \in G$, if z is a member of Z, $z = \sum_{k=1}^{n} \alpha_k T_{s_k}\mathscr{C}$, where $\alpha_k \in \mathbb{C}$ and $s_k \in G$ for $k = 1,\dots,n$, and if $t \in G$, then

$$\begin{aligned}
(C(s)z)(t) &= \mathscr{C}(s)\sum_{k=1}^{n} \alpha_k \mathscr{C}(t+s_k)\\
&= \frac{1}{2}\Big(\sum_{k=1}^{n} \alpha_k \mathscr{C}(t+s_k+s) + \sum_{k=1}^{n} \alpha_k \mathscr{C}(t+s_k-s)\Big)\\
&= \frac{1}{2}(T_s z + T_{-s} z)(t).
\end{aligned}$$

Thus Z is an invariant subspace for $C(s)$, and we have

$$C(s)\restriction_Z = \tfrac{1}{2}(\tilde{T}_s + \tilde{T}_{-s}). \tag{6.3}$$

Let $\mathfrak{B}_0$ be the subalgebra of $\mathscr{L}(Z)$ generated by the $\tilde{T}_s$'s. Obviously, $\mathfrak{B}_0$ is a unital, commutative normed algebra, with I_Z as the identity element. For each $s \in G$, let

$$\tilde{C}(s) := C(s)\restriction_Z.$$

In view of (6.3), $\tilde{C} = \{\tilde{C}(s)\}_{s\in G}$ is a cosine family in $\mathfrak{B}_0$, and, on account of (6.2),

$$\|\tilde{C}(s) - c(s)I_Z\| \le \gamma(G_{\mathrm{d}}, c) - \epsilon$$

for all $s \in G$.

STEP 3. We now work with $\tilde{C}$ and $\tilde{T}$ to obtain the main conclusion.

Let $\mathfrak{B}$ denote the completion of $\mathfrak{B}_0$, complexified if $\mathfrak{B}_0$ is real. Clearly, $\mathfrak{B}$ is a unital, commutative Banach algebra, and its identity element can be naturally identified with I_Z.

We first argue that, for each $\phi \in \Delta(\mathfrak{B})$, the mapping $G \ni s \mapsto \phi(\tilde{T}_s) \in \mathbb{C}$ is a character on G, and that

$$c(s) = \tfrac{1}{2}(\phi(\tilde{T}_s) + \phi(\tilde{T}_{-s})) \quad (s \in G). \tag{6.4}$$

Indeed, if $\phi \in \Delta(\mathfrak{B})$, then

$$\phi(\tilde{T}_{s+t}) = \phi(\tilde{T}_s \tilde{T}_t) = \phi(\tilde{T}_s)\phi(\tilde{T}_t)$$

for all $s, t \in G$, and

$$\phi(\tilde{T}_0) = \phi(I_Z) = 1.$$

Moreover, for each $s \in G$,

$$|\phi(\tilde{T}_s)| \leq \|\phi\| \, \|\tilde{T}_s\| = 1,$$

where again we used the fact that $\|\phi\| = 1$. This implies, by an argument already familiar from the proof of Theorem 5.6, that $|\phi(\tilde{T}_s)| = 1$ for every $s \in G$. Thus $s \mapsto \phi(\tilde{T}_s)$ is in fact a character on G. Applying (i) and (ii)(b) of Proposition 6.1, we now see that $s \mapsto (\phi(\tilde{T}_s) + \phi(\tilde{T}_{-s}))/2$ is a bounded cosine function. Moreover, for each $s \in G$,

$$\begin{aligned} \bigl|\tfrac{1}{2}(\phi(\tilde{T}_s) + \phi(\tilde{T}_{-s})) - c(s)\bigr| &= \bigl|\phi\bigl(\tfrac{1}{2}(\tilde{T}_s + \tilde{T}_{-s}) - c(s)I_Z\bigr)\bigr| \\ &\leq \|\phi\| \, \|\tilde{C}(s) - c(s)I_Z\| \leq \gamma(G_{\mathrm{d}}, c) - \epsilon. \end{aligned}$$

Invoking the definition of $\gamma(G_{\mathrm{d}}, c)$, we conclude that (6.4) holds.

Select, arbitrarily, $\psi \in \Delta(\mathfrak{B})$, and denote the corresponding character $s \mapsto \psi(\tilde{T}_s)$ by χ. Then, on the one hand, we have the representation of c as per (6.4) with an arbitrary character of the form $s \mapsto \phi(\tilde{T}_s)$, $\phi \in \Delta(\mathfrak{B})$, and, on the other hand, we have the representation

$$c(s) = \tfrac{1}{2}(\chi(s) + \chi(-s)) \quad (s \in G)$$

as a particular case of (6.4). Applying (ii)(a) of Proposition 6.1, we infer that the following holds: if $\phi \in \Delta(\mathfrak{B})$, then $\phi(\tilde{T}_s) = \chi(s)$ for all $s \in G$ or $\phi(\tilde{T}_s) = \chi(-s)$ for all $s \in G$.

Fix $s \in G$ arbitrarily. By the observation just made and the characterisation of the spectrum of an element of a unital, commutative Banach algebra referred to earlier,

$$\sigma_{\mathfrak{B}}(\tilde{T}_s) = \{\phi(\tilde{T}_s) \mid \phi \in \Delta(\mathfrak{B})\} \subset \{\chi(s), \chi(-s)\}.$$

Since $\tilde{T}_s$ is doubly power bounded (recall that $\|\tilde{T}_s\| = 1$ and $\|\tilde{T}_s^{-1}\| = \|\tilde{T}_{-s}\| = 1$), it follows from Proposition 6.3 that there exists an idempotent E_s in $\mathfrak{B}$ such that

$$\tilde{T}_s = \chi(s)E_s + \chi(-s)(I_Z - E_s).$$

Note in passing that, if $\sigma_{\mathfrak{B}}(\tilde{T}_s)$ consists of a single element, then E_s is either zero or equal to I_Z. Since

$$\begin{aligned} \bigl(\chi(s)E_s &+ \chi(-s)(I_Z - E_s)\bigr)\bigl(\chi(-s)E_s + \chi(s)(I_Z - E_s)\bigr) \\ &= E_s^2 + \chi(-2s)(I_Z - E_s)E_s + \chi(2s)E_s(I_Z - E_s) + (I_Z - E_s)^2 \\ &= E_s^2 + (I_Z - E_s)^2 = E_s + I_Z - E_s = I_Z, \end{aligned}$$

and since $\chi(s)E_s + \chi(-s)(I_Z - E_s)$ and $\chi(-s)E_s + \chi(s)(I_Z - E_s)$ commute (as do any other two elements of $\mathfrak{B}$), it follows that $\chi(-s)E_s + \chi(t)(I_Z - E_s)$ is the inverse of $\chi(s)E_s + \chi(-s)(I_Z - E_s)$. We thus have

$$\tilde{T}_{-s} = \chi(-s)E_s + \chi(s)(I_Z - E_s).$$

Consequently,

$$\tilde{C}(s) = \tfrac{1}{2}\big(\chi(s)E_s + \chi(-s)(I_Z - E_s) + \chi(-s)E_s + \chi(s)(I_Z - E_s)\big) = c(s)I_Z.$$

In particular, $\tilde{C}(s)\mathscr{C} = c(s)\mathscr{C}$. But, by definition of $\tilde{C}(s)$ and (6.3),

$$(\tilde{C}(s)\mathscr{C})(0) = \tfrac{1}{2}(\mathscr{C}(s) + \mathscr{C}(-s)) = \mathscr{C}(s),$$

and, independently,

$$(c(s)\mathscr{C})(0) = c(s)\mathscr{C}(0) = c(s)e_{\mathfrak{A}}.$$

Therefore $\mathscr{C}(s) = c(s)e_{\mathfrak{A}}$. Since s was arbitrary, the theorem is proved. ∎

6.3. More on γ-numbers. In this section, we present several results concerning the values of γ-numbers. Our discussion will not be as comprehensive as in the case of β-numbers, but we shall still be able to cover a wide range of cases.

We start with a lemma generalising, in one direction, conclusion (ii)(a) of Proposition 6.1.

Let G be an Abelian group. Let $\ell^\infty(G)$ denote the space of all bounded, complex functions on G, endowed with the uniform norm. A *Banach mean* on $\ell^\infty(G)$ is a complex linear functional m on $\ell^\infty(G)$ such that:

(i) $m(1_G) = 1 = \|m\|$;
(ii) $m(T_s f) = m(f)$ for each $f \in \ell^\infty(G)$ and each $s \in G$.

A familiar argument (see, e.g., [73, pp. 109–100]) shows that (i) implies

(iii) $m(f) \geq 0$ whenever $f \geq 0$ and $f \in \ell^\infty(G)$.

The concept of a Banach mean is a generalisation of that of a Banach limit [8]; see also, e.g., [19, Theorem III.7.1]. The existence of Banach means for Abelian groups was established by von Neumann [87], [88], and more generally for Abelian semigroups by Day [25]; see also [35, Theorem 1.2.1] or [38, Theorem 17.5]. Abelian groups and semigroups are examples of *amenable* groups and semigroups. A group or semigroup (discrete or topological) is called amenable if there is a Banach mean on a suitable space of bounded functions on that group or semigroup (such as $\ell^\infty(G)$ or $L^\infty(G)$ in the case of a group G) [35], [67], [68].

LEMMA 6.4. *If G is an Abelian group and if $\chi_1, \chi_2 \in \widehat{G_{\mathrm{d}}}$ are such that*

$$\|\chi_1 + \overline{\chi}_1 - (\chi_2 + \overline{\chi}_2)\|_\infty < 2,$$

then $\chi_1 = \chi_2$ or $\chi_1 = \overline{\chi}_2$.

Proof. Let m be a Banach mean on $\ell^\infty(G)$. For each $\psi \in \widehat{G_{\mathrm{d}}}$, we have

$$m(\psi) = \begin{cases} 1 & \text{if } \psi = 1_G, \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

This identity is established with a well-known argument which for convenience we reproduce here (cf. the proof of [72, Theorem 1.2.5]). The case $\psi = 1_G$ is clear. If $\psi \neq 1_G$, then $\psi(s_0) \neq 1$ for some $s_0 \in G$, and so

$$m(\psi) = m(T_{s_0}\psi) = \psi(s_0)m(\psi),$$

whence $m(\psi) = 0$.

Assume towards a contradiction that $\chi_1 \neq \chi_2$ and $\chi_1 \neq \overline{\chi}_2$. Then each of the characters $\chi_1\chi_2$, $\chi_1\overline{\chi}_2$, $\overline{\chi}_1\chi_2$, and $\overline{\chi}_1\overline{\chi}_2$ is non-trivial, and, in view of (6.5),

$$m(\chi_1\chi_2) = m(\chi_1\overline{\chi}_2) = m(\overline{\chi}_1\chi_2) = m(\overline{\chi}_1\overline{\chi}_2) = 0.$$

Since, by (6.5), $m(\psi) \geq 0$ for all $\psi \in \widehat{G_{\mathrm{d}}}$, we in particular have $m(\chi_1^2) \geq 0$, $m(\overline{\chi}_1^2) \geq 0$, $m(\chi_2^2) \geq 0$, and $m(\overline{\chi}_2^2) \geq 0$. Taking into account that

$$(\chi_1 + \overline{\chi}_1 - (\chi_2 + \overline{\chi}_2))^2 = \chi_1^2 + \overline{\chi}_1^2 + \chi_2^2 + \overline{\chi}_2^2 - 2(\chi_1\chi_2 + \chi_1\overline{\chi}_2 + \overline{\chi}_1\chi_2 + \overline{\chi}_1\overline{\chi}_2) + 4,$$

we see that

$$m\big((\chi_1 + \overline{\chi}_1 - (\chi_2 + \overline{\chi}_2))^2\big) \geq 4.$$

On the other hand,

$$m\big((\chi_1 + \overline{\chi}_1 - (\chi_2 + \overline{\chi}_2))^2\big) \leq \|(\chi_1 + \overline{\chi}_1 - (\chi_2 + \overline{\chi}_2))\|_\infty^2 < 4,$$

where the rightmost inequality holds by assumption. This contradiction establishes the lemma. ■

We are now ready to state our first, though just preliminary, result.

THEOREM 6.5. *If G is a non-zero, locally compact Abelian group and if $c \in \mathrm{Cos}(G)$, then $\gamma(G, c) \geq 1$.*

Proof. Assume to the contrary that $\gamma(G, c) < 1$. Then there exists $\tilde{c} \in \mathrm{Cos}(G)$ different from c such that $\|c - \tilde{c}\|_\infty < 1$. By (i) and (ii)(b) of Proposition 6.1, there exist χ and $\tilde{\chi}$ in $\widehat{G}$ such that

$$c = \tfrac{1}{2}(\chi + \overline{\chi}) \quad \text{and} \quad \tilde{c} = \tfrac{1}{2}(\tilde{\chi} + \overline{\tilde{\chi}}),$$

respectively. An application of Lemma 6.4 now shows that either $\chi = \tilde{\chi}$ or $\chi = \overline{\tilde{\chi}}$. In both cases $c = \tilde{c}$. This contradiction establishes the result. ■

The above theorem can be considerably strengthen based on the following result due to Esterle:

PROPOSITION 6.6. *If $c \in \mathrm{Cos}(\mathbb{Z})$, then $\gamma(\mathbb{Z}, c) \geq \sqrt{5}/2$.*

The full version of Esterle's result [31, Theorem 3.14] gives the value of $\gamma(\mathbb{Z}, c)$ for every member c of $\mathrm{Cos}(\mathbb{Z})$, and, in particular, identifies all those bounded cosine sequences c for which $\gamma(\mathbb{Z}, c) = \sqrt{5}/2$.

The generalisation of Theorem 6.5 just alluded to goes as follows.

THEOREM 6.7. *If G is a non-zero, locally compact Abelian group and if $c \in \mathrm{Cos}(G)$, then $\gamma(G, c) \geq \sqrt{5}/2$.*

Proof. If $\tilde{c} \in \mathrm{Cos}(G) \setminus \{c\}$ and if $s \in G$ is such that $\tilde{c}(s) \neq c(s)$, then $\{\tilde{c}(ns)\}_{n\in\mathbb{Z}}$ and $\{c(ns)\}_{n\in\mathbb{Z}}$ are two different cosine sequences, and we have

$$\|\tilde{c} - c\|_\infty \geq \sup_{n\in\mathbb{Z}} |\tilde{c}(ns) - c(ns)| \geq \gamma(\mathbb{Z}, \{c(ns)\}_{n\in\mathbb{Z}}).$$

Hence $\gamma(G, c) \geq \gamma(\mathbb{Z}, \{c(ns)\}_{n\in\mathbb{Z}})$, and the result now is a consequence of Proposition 6.6. ■

The following proposition is a particular case of a much more general result of Schwenninger and Zwart [76, Theorem 3.2]:

PROPOSITION 6.8. *We have* $\gamma(\mathbb{Z}, 1_{\mathbb{Z}}) = 3/2$.

For each $\omega \in \mathbb{R}$, we denote by c_ω the function $s \mapsto \cos \omega s$. It is a well-known fact that

$$\mathrm{Cos}(\mathbb{R}) = \{c_\omega \mid \omega \in \mathbb{R}\};$$

see, e.g., [2, §8, Theorem 1].

The following result has been established by Bobrowski et al. [14, Theorem 1 and Remark 1] and, independently, by Esterle [30, Lemma 3.5]:

PROPOSITION 6.9. *If* $\omega \in \mathbb{R} \setminus \{0\}$, *then* $\gamma(\mathbb{R}_{\mathrm{d}}, c_\omega) = \gamma(\mathbb{R}, c_\omega) = 8/(3\sqrt{3})$.

The following result is due to Chojnacki [18, Lemma 2] and Esterle [30, Corollary 3.7]:

PROPOSITION 6.10. *We have* $\gamma(\mathbb{R}_{\mathrm{d}}, 1_{\mathbb{R}}) = \gamma(\mathbb{R}, 1_{\mathbb{R}}) = 2$.

The rest of the section will be devoted to exploring under what conditions on G the equality $\gamma(G_{\mathrm{d}}, c) = \gamma(G, c)$ holds for all $c \in \mathrm{Cos}(G)$.

LEMMA 6.11. *Let G be a locally compact, σ-compact, divisible Abelian group. If χ is a discontinuous character on G, then every element of $\mathbb{T}$ is a limit of a net $\{\chi(s_\alpha)\}_{\alpha\in A}$, where $\{s_\alpha\}_{\alpha\in A}$ is a net in G converging to* 0.

Proof. Let C be the set of all limits of convergent nets of the form $\{\chi(s_\alpha)\}_{\alpha\in A}$, where $\{s_\alpha\}_{\alpha\in A}$ is a net in G converging to 0. If $c \in C$ and $c = \lim_{\alpha\in A} \chi(s_\alpha)$, where $\{s_\alpha\}_{\alpha\in A}$ is a net in G converging to 0, then $c^{-1} = \lim_{\alpha\in A} \chi(-s_\alpha)$ and $\{-s_\alpha\}_{\alpha\in A}$ is a net in G converging to 0. This shows that c^{-1} is in C. Suppose now that c and d are in C, and that $c = \lim_{\alpha\in A} \chi(s_\alpha)$ and $d = \lim_{\beta\in B} \chi(t_\beta)$, where $\{s_\alpha\}_{\alpha\in A}$ and $\{t_\beta\}_{\beta\in B}$ are nets in G converging to 0. Equip $A \times B$ with the product order defined by $(\alpha_1, \beta_1) \preceq (\alpha_2, \beta_2)$ whenever $\alpha_1 \preceq \alpha_2$ and $\beta_1 \preceq \beta_2$. Then $\{s_\alpha t_\beta\}_{(\alpha,\beta)\in A\times B}$ is a net in G converging to 0 and

$$\lim_{(\alpha,\beta)\in A\times B} \chi(s_\alpha t_\beta) = \lim_{(\alpha,\beta)\in A\times B} \chi(s_\alpha)\chi(t_\beta) = \lim_{\alpha\in A} \chi(s_\alpha) \lim_{\beta\in B} \chi(t_\beta) = cd.$$

This shows that cd belongs to C. We see that C is a group under multiplication. Clearly, C is also a closed subset of $\mathbb{T}$. Thus C is a closed subgroup of $\mathbb{T}$, and as such it is either finite or all of $\mathbb{T}$ (see, e.g., [61, Section 2, Corollary 3]).

Assume that the first possibility holds. Then there exists $k \in \mathbb{N}$ such that $c^k = 1$ for all $c \in C$. Fix $c \in C$ arbitrarily, and let $\{s_\alpha\}_{\alpha\in A}$ be a net in G converging to 0 such that $c = \lim_{\alpha\in A} \chi(s_\alpha)$. The homomorphism

$$m_k \colon G \to G, \qquad s \mapsto ks,$$

is continuous, and, since G is divisible, it is surjective. Since G is σ-compact, it follows from Pontryagin's open mapping theorem (see, e.g., [38, Theorem 5.29]) that m_k is open. Let $\{V_\iota\}_{\iota\in I}$ be a base of open neighbourhoods of 0 in G. Clearly, for each $\iota \in I$, $m_k(V_\iota) \cap V_\iota$ is an open neighbourhood of 0. Equip $A \times I$ with the product order, and, for each $(\alpha, \iota) \in A \times I$, pick $\beta_{\alpha,\iota} \in A$ such that $\alpha \preceq \beta_{\alpha,\iota}$ and $s_{\beta_{\alpha,\iota}} \in m_k(V_\iota) \cap V_\iota$. Then the net $\{s_{\beta_{\alpha,\iota}}\}_{(\alpha,\iota)\in A\times I}$ converges to 0 and $\lim_{(\alpha,\iota)\in A\times I} \chi(s_{\beta_{\alpha,\iota}}) = c$. Moreover, for each $(\alpha, \iota) \in A \times I$, there exists $t_{\alpha,\iota} \in V_\iota$ such that $kt_{\alpha,\iota} = s_{\beta_{\alpha,\iota}}$. Clearly, the net $\{t_{\alpha,\iota}\}_{(\alpha,\iota)\in A\times I}$ converges to 0. By the compactness of $\mathbb{T}$, there exists a subnet $\{t_\gamma\}_{\gamma\in\Gamma}$ such that the net $\{\chi(t_\gamma)\}_{\gamma\in\Gamma}$ converges. Consequently, $d = \lim_{\gamma\in\Gamma} \chi(t_\gamma)$ belongs to C. Clearly, $c = d^k$. On the other hand, d, as any other member of C, obeys $d^k = 1$, so $c = 1$. Thus $C = \{1\}$, and this implies that χ is continuous at 0, and hence everywhere on G, contrary to assumption. It follows that the other possibility concerning the form of C holds, namely $C = \mathbb{T}$. ■

LEMMA 6.12. *Let G be a locally compact, σ-compact, divisible Abelian group. If $c \in \mathrm{Cos}(G)$ and if $\tilde{c} \in \mathrm{Cos}(G_{\mathrm{d}}) \setminus \mathrm{Cos}(G)$ (so that $\tilde{c}$ is a discontinuous cosine function), then*

$$\limsup_{s \to 0} |c(s) - \tilde{c}(s)| = 2.$$

Proof. In view of (i), (ii)(b), and (ii)(c) of Proposition 6.1, there exists a discontinuous character $\tilde{\chi}$ on G such that

$$\tilde{c}(s) = \tfrac{1}{2}(\tilde{\chi}(s) + \tilde{\chi}(-s))$$

for each $s \in G$. By Lemma 6.11, there exists a net $\{s_\alpha\}_{\alpha \in A}$ in G converging to 0 such that $\lim_{\alpha \in A} \tilde{\chi}(s_\alpha) = -1$. Now, $\lim_{\alpha \in A} \tilde{\chi}(-s_\alpha) = \lim_{\alpha \in A} \tilde{\chi}(s_\alpha)^{-1} = -1$, and, as c is continuous, $\lim_{\alpha \in A} c(s_\alpha) = 1$. Hence

$$\lim_{\alpha \in A} \left| c(s_\alpha) - \tfrac{1}{2}(\tilde{\chi}(s_\alpha) + \tilde{\chi}(-s_\alpha)) \right| = |c(s_\alpha) - \tilde{c}(s_\alpha)| = 2,$$

and further

$$\limsup_{s \to 0} |c(s) - \tilde{c}(s)| \geq 2.$$

But both c and $\tilde{c}$ are bounded by 1 in modulus, so in fact

$$\limsup_{s \to 0} |c(s) - \tilde{c}(s)| = 2,$$

as was to be proved. ■

THEOREM 6.13. *Let G be a non-zero, locally compact, σ-compact, divisible Abelian group. If $c \in \mathrm{Cos}(G)$, then $\gamma(G_{\mathrm{d}}, c) = \gamma(G, c)$.*

Proof. By inspecting the definitions of $\gamma(G_{\mathrm{d}}, c)$ and $\gamma(G, c)$, we see that

$$\gamma(G_{\mathrm{d}}, c) = \min\Big(\gamma(G, c), \inf_{\tilde{c} \in \mathrm{Cos}(G_{\mathrm{d}}) \setminus \mathrm{Cos}(G)} \sup_{s \in G} |c(s) - \tilde{c}(s)|\Big).$$

By Lemma 6.12, $\sup_{s \in G} |c(s) - \tilde{c}(s)| = 2$ whenever $\tilde{c} \in \mathrm{Cos}(G_{\mathrm{d}}) \setminus \mathrm{Cos}(G)$, so the equality above reduces to

$$\gamma(G_{\mathrm{d}}, c) = \min(\gamma(G, c), 2).$$

But $\gamma(G, c) \leq 2$, and therefore $\gamma(G_{\mathrm{d}}, c) = \gamma(G, c)$. ■

6.4. Relationship with homomorphisms. In this section, we relate γ-numbers to homomorphisms from algebras of the form $L^1_{\mathrm{e}}(G)$.

Let G be a locally compact Abelian group. Given $f \in L^1_{\mathrm{e}}(G)$, the *cosine Fourier transform* of f is the function $c \mapsto \mathscr{F}^{\cos}_c(f)$ on $\mathrm{Cos}(G)$ defined by

$$\mathscr{F}^{\cos}_c(f) = \int_G f(s) c(s) \, \mathrm{d}m_G(s) \qquad (c \in \mathrm{Cos}(G)).$$

For each $c \in \mathrm{Cos}(G)$, the mapping $\mathscr{F}^{\cos}_c \colon f \mapsto \mathscr{F}^{\cos}_c(f)$ is a character on $L^1_{\mathrm{e}}(G)$. Conversely, every character on $L^1_{\mathrm{e}}(G)$ coincides with $\mathscr{F}^{\cos}_c(f)$ for some uniquely determined $c \in \mathrm{Cos}(G)$ (see, e.g., [83, Theorem 14.12]). It is readily seen that $\mathscr{F}^{\cos}_{1_G}$ is the fundamental character on $L^1_{\mathrm{e}}(G)$.

Given $s \in G$, we denote by C_s the operator on $L^1_{\mathrm{e}}(G)$ defined by

$$(C_s f)(t) = \tfrac{1}{2}(f(t+s) + f(t-s)) \quad (f \in L^1_{\mathrm{e}}(G),\ \text{a.e. } t \in G).$$

It is straightforward to verify that $\{C_s\}_{s\in G}$ is a strongly continuous cosine family on $L^1_{\mathrm{e}}(G)$ and, moreover, that

$$C_s(f \star g) = C_s f \star g = f \star C_s g$$

holds for all $s \in G$ and all $f, g \in L^1_{\mathrm{e}}(G)$.

Just like $L^1(G)$, the algebra $L^1_{\mathrm{e}}(G)$ possesses an approximate identity $\{u_\iota\}_{\iota \in I}$ satisfying $\|u_\iota\|_1 = 1$ for each $\iota \in I$. To see that this indeed is the case, select an approximate identity $\{f_\iota\}_{\iota\in I}$ for $L^1(G)$ such that $f_\iota \geq 0$ and $\|f_\iota\|_1 = 1$ for each $\iota \in I$ (cf. [63, p. 377]). For each $\iota \in I$, set

$$u_\iota(s) = \tfrac{1}{2}(f_\iota(s) + f_\iota(-s)) \quad (s \in G).$$

It is then clear that, for each $\iota \in I$, u_ι belongs to $L^1_{\mathrm{e}}(G)$ and that $\{u_\iota\}_{\iota\in I}$ is an approximate identity for $L^1_{\mathrm{e}}(G)$ such that $\|u_\iota\|_1 = 1$ for each $\iota \in I$.

The following proposition can be proved in a fashion similar to the one used to establish Proposition 3.4.

Proposition 6.14. *Let G be a locally compact Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. Let $H\colon L^1_{\mathrm{e}}(G) \to \mathfrak{A}$ be a non-zero, continuous homomorphism. Then:*

(i) *there exists a unique cosine family $\mathscr{C} = \{\mathscr{C}(s)\}_{s\in G}$ on $\mathrm{Ran}(H)$ such that*

$$\mathscr{C}(s)H(f) = H(C_s f)$$

for each $s \in G$ and each $f \in L^1_{\mathrm{e}}(G)$; the cosine family $\mathscr{C}$ is strongly continuous and satisfies $\sup_{s\in G} \|\mathscr{C}(s)\| \leq \|H\|$;

(ii) *if $\{u_\iota\}_{\iota\in I}$ is an approximate identity for $L^1_{\mathrm{e}}(G)$, then*

$$\mathscr{C}(s)x = \lim_{\iota\in I} H(C_s u_\iota)x \tag{6.6}$$

for each $s \in G$ and each $x \in \mathrm{Ran}(H)$;

(iii) *with L denoting the left regular representation of $\mathfrak{A}$, the mapping $(L \circ H)\restriction_{\mathrm{Ran}(H)}$ admits the representation*

$$\big((L \circ H)\restriction_{\mathrm{Ran}(H)}\big)(f)x =: H(f)x = \int_G f(s)\mathscr{C}(s)x \, \mathrm{d}m_G(s) \tag{6.7}$$

for each $f \in L^1_{\mathrm{e}}(G)$ and each $x \in \mathrm{Ran}(H)$.

Theorem 6.15. *Let G be a non-zero, locally compact Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. If $c \in \mathrm{Cos}(G)$ and if $H\colon L^1_{\mathrm{e}}(G) \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}^{\cos}_c\| < \gamma(G_{\mathrm{d}}, c)$, then there exists an idempotent e in $\mathfrak{A}$ such that $H = e \otimes \mathscr{F}^{\cos}_c$.*

Proof. As might be expected, the proof of this theorem is closely patterned on the proof of Theorem 3.5.

Step 1. If $H = 0$, then $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}^{\cos}_c\| < \gamma(G_{\mathrm{d}}, c)$, since $\|\mathscr{F}^{\cos}_c\| = \|c\|_\infty = 1$ implies that $\|e_{\mathfrak{A}} \otimes \mathscr{F}^{\cos}_c\| = \|e_{\mathfrak{A}}\|\,\|\mathscr{F}^{\cos}_c\| = 1$ and since $\gamma(G_{\mathrm{d}}, c) \geq \sqrt{5}/2$ by Theorem 6.7. Also, if $H = 0$, then $H = 0 \otimes \mathscr{F}_\chi$, where the last 0 is the zero element of $\mathfrak{A}$. Thus the theorem holds when H is zero. Suppose from now on that H is non-zero. Choose a contractive approximate identity $\{u_\iota\}_{\iota\in I}$ for $L^1_{\mathrm{e}}(G)$. Let $\{\mathscr{C}(s)\}_{s\in G}$ be the cosine family on $\mathrm{Ran}(H)$ whose existence is guaranteed by Proposition 6.14. If $s\in G$, if $x\in\mathrm{Ran}(H)$, and if $\iota\in I$, then

$$\begin{aligned}\|H(C_s u_\iota)x-(e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos})(C_s u_\iota)x\| &\le \|H-e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos}\|\,\|C_s u_\iota\|_1\|x\|\\ &\le \|H-e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos}\|\,\|x\|.\end{aligned}\tag{6.8}$$

Since c is continuous, we have $\lim_{\iota\in I}\mathscr{F}_c^{\cos}(C_s u_\iota)=c(s)$, and so

$$\lim_{\iota\in I}(e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos})(C_s u_\iota)x=c(s)x.$$

Combining this with (6.6), we deduce from (6.8) that

$$\|\mathscr{C}(s)x-c(s)x\|\le\|H-e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos}\|\,\|x\|.$$

Hence

$$\sup_{s\in G}\|\mathscr{C}(s)-c(s)I_{\mathrm{Ran}(H)}\|\le\|H-e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos}\|.$$

This together with the assumption that $\|H-e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos}\|<\gamma(G_{\mathrm{d}},c)$ yields

$$\sup_{s\in G}\|\mathscr{C}(s)-c(s)I_{\mathrm{Ran}(H)}\|<\gamma(G_{\mathrm{d}},c).$$

An application of Theorem 6.2 now implies that $\mathscr{C}(s)=c(s)I_{\mathrm{Ran}(H)}$ for all $s\in G$, and further, by (6.7), that

$$H(f)x=\mathscr{F}_c^{\cos}(f)x\tag{6.9}$$

for all $f\in L^1_{\mathrm{e}}(G)$ and all $a\in\mathrm{Ran}(H)$.

STEP 2. If $f\in L^1_{\mathrm{e}}(G)$, then

$$H(f)=\lim_{\iota\in I}H(u_\iota)H(f)=\lim_{\iota\in I}H(f)H(u_\iota).\tag{6.10}$$

For each $\iota\in I$, putting $x=H(u_\iota)$ in (6.9) yields

$$H(f)H(u_\iota)=\mathscr{F}_c^{\cos}(f)H(u_\iota).$$

Combining this with (6.10), we find that

$$H(f)=\lim_{\iota\in I}\mathscr{F}_c^{\cos}(f)H(u_\iota).\tag{6.11}$$

Now, arguing as in the proof of Theorem 3.5, we conclude that the limit $\lim_{\iota\in I}H(u_\iota)$ exists and is an idempotent in $\mathrm{Ran}(H)$. This together with (6.11) establishes the theorem. ∎

6.5. Applications. We now combine together various results from the last and previous sections to draw conclusions regarding isolability properties of homomorphisms.

The following theorem is an immediate consequence of Theorems 6.7 and 6.15.

THEOREM 6.16. *Let G be a non-zero, locally compact Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. If $c\in\mathrm{Cos}(G)$ and if $H\colon L^1_{\mathrm{e}}(G)\to\mathfrak{A}$ is a continuous homomorphism such that $\|H-e_{\mathfrak{A}}\otimes\mathscr{F}_c^{\cos}\|<\sqrt{5}/2$, then there exists an idempotent e in $\mathfrak{A}$ such that $H=e\otimes\mathscr{F}_c^{\cos}$.*

Combining Theorem 6.16 with Theorem 2.16 and the fact that $\|\mathscr{F}_c^{\cos}\| = 1$ for each $c \in \mathrm{Cos}(G)$, we obtain the following theorem.

THEOREM 6.17. *Let G be a non-zero, locally compact Abelian group, and let $\mathfrak{A}$ be a unital normed algebra. If $c \in \mathrm{Cos}(G)$, then $e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos} \colon L^1_{\mathrm{e}}(G) \to \mathfrak{A}$ is totally isolated.*

We also immediately deduce the following result (cf. the proof of Theorem 5.23).

THEOREM 6.18. *Let G be a non-zero, locally compact Abelian group, and let $\mathfrak{A} \in \mathscr{A}_{\mathrm{wni}}$. If $c \in \mathrm{Cos}(G)$ and if $H \colon L^1_{\mathrm{e}}(G) \to \mathfrak{A}$ is a non-zero homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos}\| < \gamma(G_{\mathrm{d}}, c)$, then $H = e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos}$.*

Applying Theorem 6.18 with c equal to 1_G, we obtain the following theorem.

THEOREM 6.19. *If G is a non-zero, locally compact Abelian group, then $\gamma(G_{\mathrm{d}}, 1_G) \le \alpha(L^1_{\mathrm{e}}(G))$.*

Combining this theorem with Theorem 6.7 yields the following corollary.

COROLLARY 6.20. *If G is a non-zero, locally compact Abelian group, then $\alpha(L^1_{\mathrm{e}}(G)) \ge \sqrt{5}/2$.*

The following theorem is a partial converse to Theorem 6.19.

THEOREM 6.21. *If G is a non-zero, locally compact Abelian group, then $\alpha(L^1_{\mathrm{e}}(G)) \le \gamma(G, 1_G)$.*

Proof. The proof is similar to that of Theorem 5.26. For each $c \in \mathrm{Cos}(G)$, $\mathscr{F}_c^{\cos}$ is a non-zero homomorphism from $L^1_{\mathrm{e}}(G)$ into $\mathbb{C}$ and

$$\|\mathscr{F}_c^{\cos} - \mathscr{F}_{1_G}^{\cos}\| = \sup_{s \in G} |c(s) - 1|. \tag{6.12}$$

Moreover, if $c \ne 1_G$, then $\mathscr{F}_c^{\cos} \ne \mathscr{F}_{1_G}^{\cos}$. Since $\mathscr{F}_{1_G}^{\cos}$ is the fundamental character on $L^1_{\mathrm{e}}(G)$ and since $\mathbb{C}$ is an algebra with no non-trivial idempotents, it follows that

$$\alpha(L^1(G)) \le \inf_{c \in \mathrm{Cos}(G) \setminus \{1_G\}} \|\mathscr{F}_c^{\cos} - \mathscr{F}_{1_G}^{\cos}\|.$$

In view of (6.12), this implies that $\alpha(L^1_{\mathrm{e}}(G)) \le \gamma(G, 1_G)$. ■

Combining Theorems 6.19 and 6.21 leads immediately to the following result.

THEOREM 6.22. *If G is a non-zero, locally compact Abelian group such that $\gamma(G_{\mathrm{d}}, 1_G) = \gamma(G, 1_G)$, then $\alpha(L^1_{\mathrm{e}}(G)) = \gamma(G, 1_G)$.*

In turn, the following theorem is an immediate consequence of Theorems 6.13 and 6.22.

THEOREM 6.23. *If G is a non-zero, locally compact, σ-compact, divisible Abelian group, then $\alpha(L^1_{\mathrm{e}}(G)) = \gamma(G, 1_G)$.*

Since, for a non-zero, discrete Abelian group G, there is no distinction between $\gamma(G_{\mathrm{d}}, 1_G)$ and $\gamma(G, 1_G)$, we readily obtain the following theorem.

THEOREM 6.24. *If G is a non-zero, discrete Abelian group, then $\alpha(\ell^1_{\mathrm{e}}(G)) = \gamma(G, 1_G)$.*

COROLLARY 6.25. *We have:*

(i) $\alpha(\ell^1_{\mathrm{e}}(\mathbb{Z})) = 3/2$;
(ii) $\alpha(\ell^1_{\mathrm{e}}(\mathbb{R})) = 2$;
(iii) $\alpha(L^1_{\mathrm{e}}(\mathbb{R})) = 2$.

Proof. Statement (i) follows from Proposition 6.8 and Theorem 6.24, statement (ii) follows from Proposition 6.10 and Theorem 6.24, and statement (iii) follows from Proposition 6.10 and Theorem 6.22. ■

Theorem 6.15 and Propositions 6.9 and 6.10 yield the following result.

THEOREM 6.26. *Let $\mathfrak{A}$ be a unital normed algebra. If $c \in \mathrm{Cos}(\mathbb{R})$ and if $H\colon L^1_{\mathrm{e}}(\mathbb{R}) \to \mathfrak{A}$ is a continuous homomorphism such that*

$$\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos}\| < \begin{cases} \frac{8}{3\sqrt{3}} & \text{if } c \neq 1_{\mathbb{R}}, \\ 2 & \text{if } c = 1_{\mathbb{R}}, \end{cases}$$

then there exists an idempotent e in $\mathfrak{A}$ such that $H = e \otimes \mathscr{F}_c^{\cos}$.

REMARK 6.27. In general, '$<$' in Theorem 6.26 cannot be replaced by '$\leq$'. To see that this indeed is so, suppose that $\mathfrak{A}$ is a unital normed algebra with a non-trivial idempotent f of norm 1 (cf. Remark 2.8). Consider first the case $c \neq 1_{\mathbb{R}}$. Then $c = c_\omega$ for some $\omega \in \mathbb{R} \setminus \{0\}$. Let $H\colon L^1_{\mathrm{e}}(\mathbb{R}) \to \mathfrak{A}$ be the homomorphism given by

$$H = (e_{\mathfrak{A}} - f) \otimes \mathscr{F}_{c_\omega}^{\cos} + f \otimes \mathscr{F}_{c_{3\omega}}^{\cos}.$$

Since

$$H - e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos} = f \otimes (\mathscr{F}_{c_{3\omega}}^{\cos} - \mathscr{F}_{c_\omega}^{\cos}),$$

it follows that

$$\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos}\| = \|f\|\,\|\mathscr{F}_{c_{3\omega}}^{\cos} - \mathscr{F}_{c_\omega}^{\cos}\| = \sup_{t\in\mathbb{R}} |\cos 3\omega t - \cos \omega t| = \frac{8}{3\sqrt{3}};$$

see [14, Lemma 1] for the rightmost equality. This together with the fact that H is not of the form $e \otimes \mathscr{F}_c^{\cos}$, where e is an idempotent of $\mathfrak{A}$, shows that '$<$' in the upper inequality in Theorem 6.26 cannot be replaced by '$\leq$'.

Consider now the case $c = 1_{\mathbb{R}}$. Let $H\colon L^1_{\mathrm{e}}(\mathbb{R}) \to \mathfrak{A}$ be the homomorphism given by

$$H = (e_{\mathfrak{A}} - f) \otimes \mathscr{F}_{1_{\mathbb{R}}}^{\cos} + f \otimes \mathscr{F}_{c_1}^{\cos}.$$

Since

$$H - e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos} = f \otimes (\mathscr{F}_{c_1}^{\cos} - \mathscr{F}_{1_{\mathbb{R}}}^{\cos}),$$

it follows that

$$\|H - e_{\mathfrak{A}} \otimes \mathscr{F}_c^{\cos}\| = \|f\|\,\|\mathscr{F}_{c_1}^{\cos} - \mathscr{F}_{1_{\mathbb{R}}}^{\cos})\| = \sup_{t\in\mathbb{R}} |\cos t - 1| = 2.$$

This together with H not being of the form $e \otimes \mathscr{F}_c^{\cos}$, where e is an idempotent of $\mathfrak{A}$, shows in turn that '$<$' in the lower inequality in Theorem 6.26 cannot be replaced by '$\leq$'. The argument is complete.

Let $\mathfrak{X}$ be a complex normed algebra such that $\Delta(\mathfrak{X}) \neq \emptyset$. For each $\delta > 0$, we introduce the following condition on $\mathfrak{X}$:

(C_δ) if $\mathfrak{A}$ is a unital normed algebra, if $\phi \in \Delta(\mathfrak{X})$, and if $H\colon \mathfrak{X} \to \mathfrak{A}$ is a continuous homomorphism such that $\|H - e_{\mathfrak{A}} \otimes \phi\| < \delta$, then there exists an idempotent e in $\mathfrak{A}$ such that $H = e \otimes \phi$.

A straightforward argument yields the following result.

THEOREM 6.28. *For each $\delta > 0$, satisfaction of* (C_δ) *is invariant under isomorphic isomorphisms of normed algebras.*

We are now in a position to prove our final theorem.

THEOREM 6.29.

(i) $L^1(\mathbb{R})$ *and* $L^1_{\mathrm{e}}(\mathbb{R})$ *are not isometrically isomorphic as normed algebras.*
(ii) $\ell^1(\mathbb{R})$ *and* $\ell^1_{\mathrm{e}}(\mathbb{R})$ *are not isometrically isomorphic as normed algebras.*

Proof. (i) Since $\beta(\mathbb{R}_{\mathrm{d}}) = 2$ by Theorem 5.15, we can apply Theorem 5.20 to conclude that $L^1(\mathbb{R})$ satisfies (C_2). Let $\mathfrak{A}$ be a unital normed algebra with a non-trivial idempotent f of norm 1. Choose $\omega \in \mathbb{R} \setminus \{0\}$ and let $H\colon L^1_{\mathrm{e}}(\mathbb{R}) \to \mathfrak{A}$ be the homomorphism given by

$$H = (e_{\mathfrak{A}} - f) \otimes \mathscr{F}^{\cos}_{c_\omega} + f \otimes \mathscr{F}^{\cos}_{c_{3\omega}}.$$

The argument from Remark 6.27 shows that

$$\|H - e_{\mathfrak{A}} \otimes \mathscr{F}^{\cos}_{c_\omega}\| = \frac{8}{3\sqrt{3}}.$$

Since H is not of the form $e \otimes \mathscr{F}^{\cos}_{c_\omega}$, where e is an idempotent of $\mathfrak{A}$, it follows that $L^1_{\mathrm{e}}(\mathbb{R})$ does not satisfy (C_δ) for any $\delta > 8/(3\sqrt{3})$, and, in particular, that $L^1_{\mathrm{e}}(\mathbb{R})$ does not satisfy (C_2). Now, to complete the argument, it suffices to invoke Theorem 6.28.

(ii) The proof mimics closely the proof of the preceding statement. First, Theorems 5.15 and 5.20 ensure, like before, that $\ell^1(\mathbb{R})$ satisfies (C_2). Second, if $\mathfrak{A}$ is a unital normed algebra with a non-trivial idempotent f of norm 1, if ω is a non-zero real number, and if $H\colon \ell^1_{\mathrm{e}}(\mathbb{R}) \to \mathfrak{A}$ is the homomorphism given by

$$H = (e_{\mathfrak{A}} - f) \otimes \mathscr{F}^{\cos}_{c_\omega} + f \otimes \mathscr{F}^{\cos}_{c_{3\omega}},$$

where $\mathscr{F}^{\cos}_{c_\omega}$ and $\mathscr{F}^{\cos}_{c_{3\omega}}$ are this time viewed as characters on $\ell^1_{\mathrm{e}}(\mathbb{R})$, then, as earlier,

$$\|H - e_{\mathfrak{A}} \otimes \mathscr{F}^{\cos}_{c_\omega}\| = \frac{8}{3\sqrt{3}}.$$

This implies that $\ell^1_{\mathrm{e}}(\mathbb{R})$ does not satisfy (C_δ) for any $\delta > 8/(3\sqrt{3})$, and, in particular, that $\ell^1_{\mathrm{e}}(\mathbb{R})$ does not satisfy (C_2). An appeal to Theorem 6.28 now finishes the proof. ■

7. Closure

We close by offering a kind of symbolic summary of the developments presented in the memoir. For this we detail nine ordered AL-algebras, no two of which are isometrically algebra and order isomorphic. The algebras are listed in the first column of Table 2. The remaining columns contain attributes enabling unique identification of the isomorphism type of each of the algebras listed. Finding differences between various isomorphism types is straightforward.

Table 2. Comparison of nine ordered AL-algebras

	separable	unital	α	(C_2)
$\ell^1(\mathbb{Z}^+)$	$+$	$+$	1	
$\ell^1(\mathbb{Z})$	$+$	$+$	$\sqrt{3}$	
$\ell^1_{\mathrm{e}}(\mathbb{Z})$	$+$	$+$	3/2	
$\ell^1(\mathbb{R}^+)$	$-$	$+$	1	
$\ell^1(\mathbb{R})$	$-$	$+$	2	$+$
$\ell^1_{\mathrm{e}}(\mathbb{R})$	$-$	$+$	2	$-$
$L^1(\mathbb{R}^+)$	$+$	$-$	1	
$L^1(\mathbb{R})$	$+$	$-$	2	$+$
$L^1_{\mathrm{e}}(\mathbb{R})$	$+$	$-$	2	$-$

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ERRATA

Page, line	For	Read
23_{11}	Theorem 11.31	Theorem 11.32
24_7	$\|u_\iota\|$	$\|u_\iota\|_1$
24_2	$\|u_\iota\|$	$\|u_\iota\|_1$
28_1	$I_{\mathfrak{B}}$	$I_{\mathrm{Ran}(H)}$
29^7	$*$	$\star$
29^{20}, in two instances	$*$	$\star$
29^{21}, in two instances	$*$	$\star$
38_7, in three instances	$*$	$\star$
39_6, in two instances	$*$	$\star$
39_1	$*$	$\star$

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