ON SOME DISTAL FUNCTIONS

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1. Introduction

Let S be a discrete Abelian semigroup, $l^\infty(S)$ the space of all complex bounded functions on S endowed with the topology of pointwise convergence, and $\mathcal{L}(l^\infty(S))$ the space of all linear operators in $l^\infty(S)$ equipped with the corresponding strong operator topology. For each $s \in S$, let T_s be the operator of translation by s, defined by

$$(T_s f)(t) = f(s+t) \quad (f \in l^\infty(S),\ t \in S).$$

Let $\Sigma(S)$ be the closure of $\{T_s : s \in S\}$ in $\mathcal{L}(l^\infty(S))$.

An element f of $l^\infty(S)$ is called a *distal function* on S if the following conditions are satisfied:

(i) $f = \sigma(f)$ for some σ in $\Sigma(S)$;
(ii) if $\sigma\tau_1(f) = \sigma\tau_2(f)$ for σ, τ_1, τ_2 in $\Sigma(S)$, then

$$\tau_1(f) = \tau_2(f).$$

Namioka [**4**] established a criterion for distality with the use of which he proved the following generalization of a result of Knapp [**3**]:

If S is a subsemigroup of the group of integers and p is a real polynomial, then the function f on S defined by

$$f(n) = e^{ip(n)} \quad (n \in S)$$

is distal.

In this note, we establish one more theorem from which the latter result may easily be deduced. Its proof will be straightforward and will make no appeal to rather involved arguments of Knapp and Namioka.

2. Preliminaries

If A is a subset of the domain of a function f, we denote by $f|_A$ the restriction of f to A.

A complex function with values of unit modulus will be called *unitary*.

Let f be a unitary function on S. For each $s \in S$, put

$$\delta_s f = \bar{f} \cdot T_s f$$

and, for any $s_1, \ldots, s_n \in S$, set inductively

$$\delta_{s_1 \ldots s_n} f = \delta_{s_n}(\delta_{s_1 \ldots s_{n-1}} f).$$

A *Banach mean* on $l^\infty(S)$ is a linear functional m on $l^\infty(S)$, continuous under the supremum norm on $l^\infty(S)$, satisfying the following conditions:

(a) $\|m\| = 1 = m(1)$;
(b) $m(T_s f) = m(f)$ for each $f \in l^\infty(S)$ and each $s \in S$.

The existence of the Banach mean on $l^\infty(S)$ is ensured by a theorem of Day [**1**].

Let G be a discrete Abelian group. An element of $l^\infty(G)$ is called an *almost periodic function* on G if the set $\{T_g f\colon g \in G\}$ is relatively compact under the supremum norm on $l^\infty(G)$. The set $\mathrm{AP}(G)$ of all almost periodic cost functions on G is a subalgebra of $l^\infty(G)$, closed in norm and under conjugation of functions. If f is an almost periodic function on G and m is a Banach mean on $l^\infty(G)$, then $m(f)$ does not depend on the particular choice of m, and setting

$$(f|g) = m(f\bar{g}) \quad (f, g \in \mathrm{AP}(G))$$

defines a scalar product on $\mathrm{AP}(G)$. We shall denote by $|||\cdot|||$ the norm derived from that scalar product.

3. A distality condition

We shall find it convenient to reformulate the distality condition given in the Introduction.

Let I be the identity operator in $l^\infty(S)$ and let

$$\varSigma^*(S) = \varSigma(S) \cup \{I\}.$$

Proposition. *An element f of $l^\infty(S)$ is a distal function on S if and only if the following condition is satisfied:*

($*$) *if $\sigma\tau_1(f) = \sigma\tau_2(f)$ for σ in $\varSigma(S)$ and τ_1, τ_2 in $\varSigma^*(S)$, then*

$$\tau_1(f) = \tau_2(f).$$

Proof. That (i) and (ii) imply ($*$) is evident. Also the implication ($*$) $\Rightarrow$ (ii) is clear. To establish ($*$) $\Rightarrow$ (i), note first that $\varSigma(S)$ is a right topological semigroup, that is, for each $\tau \in \varSigma(S)$, the mapping

$$\varSigma(S) \ni \sigma \mapsto \sigma\tau \in \varSigma(S)$$

is continuous. Moreover, by Tikhonov's theorem, $\varSigma(S)$ is compact. Now a lemma of Ellis [**2**] ensures the existence of an idempotent ϵ in $\varSigma(S)$. The identity $\epsilon(f) = \epsilon(\epsilon(f))$ jointly with ($*$) implies that $f = \epsilon(f)$. The proof is complete.

4. The result

Our major result is the following

Theorem. *Let S be a subsemigroup of a discrete Abelian group G. Suppose that f is a unitary function on G such that, for some non-negative n and all $s_1, \ldots, s_n \in S$, the function $\delta_{s_1 \ldots s_n} f$ is almost periodic (when $n = 0$, we assume that the latter function coincides with f). Then the restriction of f to S is distal.*

Proof. Since the restriction to a subgroup of an almost periodic function defined on a group is almost periodic, without loss of generality we may assume that G coincides with the group

$$S - S = \{g \in G\colon g = s_1 - s_2 \text{ for } s_1, s_2 \in S\}.$$

Suppose that

$$\sigma\tau_1(f|_S) = \sigma\tau_2(f|_S) \tag{1}$$

for $\sigma \in \varSigma(S)$ and $\tau_1, \tau_2 \in \varSigma^*(S)$. Let μ be a Banach mean on $l^\infty(S)$. Define a linear functional m on $l^\infty(G)$ by setting

$$m(h) = \mu(\sigma(h|_S)) \quad (h \in l^\infty(G)).$$

We claim that m is a Banach mean on $l^\infty(G)$.

Indeed, we clearly have $\|m\| = 1 = m(1)$. If $g = s_1 - s_2$ with s_1 and s_2 in S, and $h \in l^\infty(G)$, then

$$m(T_g h) = \mu(T_{s_2}(\sigma(T_g h|_S))) = \mu(T_{s_1}(\sigma(h|_S))) = m(h),$$

which establishes the claim.

Given $s_1, \ldots, s_n \in S$, we have

$$\sigma\tau_1(\delta_{s_1\ldots s_n} f|_S) = \delta_{s_1\ldots s_n}\sigma\tau_1(f|_S) = \delta_{s_1\ldots s_n}\sigma\tau_2(f|_S) = \sigma\tau_2(\delta_{s_1\ldots s_n} f|_S). \tag{2}$$

A moment's reflection shows that there exist $\bar\tau_1$ and $\bar\tau_2$ in $\Sigma^*(G)$ such that

$$\bar\tau_i(\delta_{s_1\ldots s_n} f)|_S = \tau_i(\delta_{s_1\ldots s_n} f|_S) \quad (i = 1, 2).$$

The functions $\bar\tau_i(\delta_{s_1\ldots s_n} f)$ $(i = 1, 2)$ are clearly unitary almost periodic, so the function

$$u = \bar\tau_1(\delta_{s_1\ldots s_n} f)[\bar\tau_2(\delta_{s_1\ldots s_n} f)]^{-1}$$

is also unitary almost periodic. In view of (2),

$$m(u) = 1. \tag{3}$$

Rewriting the latter identity in the form

$$(u|1) = |||u|||\,|||1|||,$$

we see that u is constant. Now it results from (3) that actually $u = 1$, whence

$$\delta_{s_1\ldots s_n}\tau_1(f|_S) = \delta_{s_1\ldots s_n}\tau_2(f|_S). \tag{4}$$

Put

$$v = \tau_1(f|_S)[\tau_2(f|_S)]^{-1}.$$

The function v is unitary and, by (4), we have

$$\delta_{s_1\ldots s_n} v = 1 \quad \text{for any } s_1, \ldots, s_n \in S.$$

Since any unitary function e on S with $\delta_s e = 1$ for each $s \in S$ is constant, the function $\delta_{s_1\ldots s_{n-1}} v$ is constant for any $s_1, \ldots, s_{n-1} \in S$. By (1), we have $\sigma(v) = 1$, therefore

$$\delta_{s_1\ldots s_{n-1}} v = \sigma(\delta_{s_1\ldots s_{n-1}} v) = \delta_{s_1\ldots s_{n-1}}\sigma(v) = 1.$$

The repeated use of this argument shows that $v = 1$, whence

$$\tau_1(f|_S) = \tau_2(f|_S).$$

To complete the proof, it suffices now to apply the Proposition.

References

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