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Group representations of bounded cosine functions

Dedicated to Professor Jan Kisyński

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1. Introduction

Let G be a locally compact Abelian group with group operation denoted additively, and let A be an algebra with identity over the field $\mathbb{C}$ of complex numbers. A *homomorphism* from G into A is a mapping $\mathscr{G} : G \to A$ satisfying both Cauchy's functional equation

$$\mathscr{G}(a+b) = \mathscr{G}(a)\mathscr{G}(b) \quad (a, b \in G)$$

and the condition

$$\mathscr{G}(0) = e,$$

where 0 denotes the neutral element of G and e denotes the identity of A. An A-valued *cosine function* on G is a mapping $\mathscr{C} : G \to A$ satisfying both d'Alembert's functional equation

$$(1.1) \qquad \mathscr{C}(a+b) + \mathscr{C}(a-b) = 2\mathscr{C}(a)\mathscr{C}(b) \quad (a, b \in G)$$

and the condition

$$\mathscr{C}(0) = e.$$

A cosine function $\mathscr{C} : G \to A$ has a *group representation* if there is a homomorphism $\mathscr{G} : G \to A$ such that, for each $a \in G$,

$$(1.2) \qquad \mathscr{C}(a) = \frac{1}{2}\left(\mathscr{G}(a) + \mathscr{G}(-a)\right).$$

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A *semitopological algebra* is an algebra equipped with a topology such that the underlying vector space is locally convex and Hausdorff, and the multiplication operation is separately continuous. Henceforth we shall assume that A is a semitopological algebra with identity. If $\mathscr{C}: G \to A$ is a continuous cosine function such that (1.2) holds for a continuous homomorphism $\mathscr{G}: G \to A$, we say that $\mathscr{C}$ has a regular group representation. We recall that a mapping with values in a topological vector space is bounded if the image of this mapping is bounded. If $\mathscr{C}: G \to A$ is a bounded (continuous) cosine function such that (1.2) holds for a bounded (continuous) homomorphism $\mathscr{G}: G \to A$, then $\mathscr{C}$ will be said to have a bounded (regular) group representation. Given a locally convex vector space E, we denote by $\mathscr{L}(E)$ the algebra of all linear continuous operators in E. We shall always consider $\mathscr{L}(E)$ as being equipped with the strong operator topology. Under this topology, $\mathscr{L}(E)$ is a semitopological algebra. A homomorphism from G into $\mathscr{L}(E)$ will often be referred to as a representation of G in E. An $\mathscr{L}(E)$-valued cosine function will at times be referred to as a cosine function in E. Note that if E is either a Banach space or the dual of a Banach space under the $*$-weak topology, and if f is a function from a set S into $\mathscr{L}(E)$, then, as recourse to the Banach-Steinhaus theorem shows, f is bounded if and only if it is bounded in norm; that is, $\sup_{s \in S} \| f(s) \| < +\infty$.

This paper deals with the problem of group representability of cosine functions. In the context of operator-valued cosine functions, the problem arose out of a study of the Cauchy problem for second order abstract differential equations. Such a study is greatly facilitated when suitable cosine functions defined on the group $\mathbb{R}$ of real numbers and taking on values in $\mathscr{L}(E)$, where E is a Banach space, have a regular group representation (cf. [8], Section 2.5, [9], Section III.6, [16], Section III.1.1, [18]). In general, unbounded continuous cosine functions, even as simple as those taking on values in finite-dimensional algebras, may fail to admit a regular group representation; for example, the $\mathscr{L}(\mathbb{C}^2)$-valued cosine function on $\mathbb{R}$ given by

$$t \mapsto \begin{pmatrix} 1 & t^2/2 \\ 0 & 1 \end{pmatrix}$$

has no regular group representation (cf. [14]). Contrasting with this is a result of [3] stating that every $\mathscr{L}(H)$-valued bounded continuous cosine function on G, H being a Hilbert space and G being a locally compact Abelian group, has a regular group representation. Here the appeal to a Hilbert space structure is essential. In fact, Kisyński [14], [15] and Fattorini [7] (see also [9], Section III.8) exhibited various $\mathscr{L}(E)$-valued bounded cosine functions on $\mathbb{R}$ that have no regular group representation, where E are Banach spaces consisting of odd functions or of odd measures on $\mathbb{R}$ equipped either with the norm topology or with the $*$-weak topology.

In the development that follows, we characterise – in purely algebraic terms – the class of all locally compact Abelian groups G such that every bounded continuous cosine function on G with values in a sequentially complete semitopological algebra with identity has a regular group representation. We also characterise the class of all locally compact Abelian groups G such that every bounded cosine function on G with values in a sequentially complete semitopological algebra with identity has a *bounded* regular group representation. It will follow from these characterisations that the group $\mathbb{Z}$ of integers is in the former class but not in the latter. It will also become apparent that $\mathbb{R}$ is in none of the two classes. The latter result (which, of course, can also immediately be deduced from the above-mentioned results of Kisyński and Fattorini) will be a consequence of the existence of a number of $\mathscr{L}(E)$-valued bounded continuous cosine functions on $\mathbb{R}$ admitting no regular group representation, where E are Banach spaces consisting of *even* functions or of *even* measures on $\mathbb{R}$ equipped either with the norm topology or with the $*$-weak topology. Here the appeal to even functions and measures is representative of the approach adopted in handling arbitrary locally compact Abelian groups. It turns out that the use of such functions and measures renders the analysis of certain crucial cosine functions a relatively easy task.

The remainder of the paper is organised as follows. Section 2 introduces the concepts of a c-group and of a bc-group, and reduces the main representability problem to that of characterisation of the classes comprising all c-groups and all bc-groups. Section 3 gives necessary conditions for a given group to be a c-group or to be a bc-group. Section 4 is concerned with compact Abelian groups that are decomposable in a certain sense. The results of that section are used in Section 5 to demonstrate that the sufficient conditions from Section 3 are also necessary. Finally, Section 6 indicates some applications of the results obtained to a study of single operators in Banach spaces.

2. Group representations

In this section, we present our main results concerning group representability of bounded cosine functions. We show that the representability problem can be reduced to a question from harmonic analysis. Being of independent interest, this question will be the subject of a detailed analysis in the subsequent sections. We begin our considerations by introducing some notation and terminology.

Let G be an Abelian group. For subsets A and B of G, and an element a of G, we let

$$-A = \{g \in G\colon -g \in A\},$$

$$A + a = \{g \in G\colon g - a \in A\},$$

$$A + B = \{g \in G\colon g = a + b \text{ for } a \in A \text{ and } b \in B\}.$$

For $n \in \mathbb{N}$ with $n \geqq 2$, we denote by m_n the homomorphism from G into itself given by

$$m_n(a) = na \quad (a \in G).$$

We designate by $G^{(n)}$ and $G_{(n)}$ the image and kernel of m_n, respectively. If H is an Abelian group, we use $G \simeq H$ to indicate that G and H are isomorphic. If G and H are locally compact, we write $G \cong H$ to indicate that G and H are topologically isomorphic. If H is a subgroup of G, then G/H denotes the corresponding quotient group. If G is locally compact and H is closed, then G/H will be assumed to have the quotient topology under which this quotient group is locally compact. If $\{G_i\}_{i\in I}$ is an indexed collection of Abelian groups, we write $\prod_{i\in I} G_i$ for the direct product of the G_i. If $I = \{1, \ldots, n\}$, we also write $G_1 \times \cdots \times G_n$ in place of $\prod_{i\in I} G_i$. The subgroup of all sequences $\{x_i\}_{i\in I}$ in $\prod_{i\in I} G_i$ such that $x_i = 0$ for all but finitely many $i \in I$ is written $\prod^*_{i\in I} G_i$ and is referred to as the weak direct product of the groups G_i. If $\mathfrak{m}$ is a cardinal number and if, for some fixed G, $G_i = G$ for each $i \in I$, where I is a set of cardinality equal to $\mathfrak{m}$, we write $G^{\mathfrak{m}}$ for $\prod_{i \in I} G_i$ and $G^{\mathfrak{m}*}$ for $\prod_{i \in I}^{*} G_i$. Direct products of locally compact Abelian groups will be assumed to have the product topology. We recall that, for an indexed collection $\{G_i\}_{i \in I}$ of locally compact Abelian groups, $\prod_{i \in I} G_i$ is a locally compact Abelian group if an only if G_i is compact for all but finitely many $i \in I$.

Hereafter we shall assume that G is a locally compact Abelian group. We shall denote by $\mathfrak{B}(G)$ the σ-algebra of all Borel subsets of G. If $a \in G$, then δ_a will stand for the Dirac measure on G concentrated at a. We denote by $M(G)$ the space of all complex bounded regular Borel measures on G. With the norm $\|\mu\| = |\mu|(G)$, where $|\mu|$ denotes the total variation of the measure $\mu \in M(G)$, $M(G)$ is a Banach space. $M(G)$ is also a Banach algebra with identity under the convolution multiplication

$$(\mu * \nu)(E) = \int_G \mu(E - a)\, d\nu(a) \quad (\mu, \nu \in M(G),\ E \in \mathfrak{B}(G))$$

and with δ_0 as identity. We designate by $M_a(G)$ the Banach algebra of all atomic measures in $M(G)$. A measure $\mu \in M(G)$ will be termed *even* if

$$\mu(E) = \mu(-E) \quad (E \in \mathfrak{B}(G)).$$

The even measures belonging to $M(G)$ form a Banach algebra that will be denoted by $M_{\mathrm{e}}(G)$. $M_{\mathrm{ae}}(G)$ will signify the Banach algebra of all atomic measures in $M_{\mathrm{e}}(G)$.

There exists a natural cosine function $\mathscr{C}$ on G taking on values in $M_{\mathrm{ae}}(G)$ defined by

$$\mathscr{C}(a) = \frac{1}{2}(\delta_a + \delta_{-a}) \quad (a \in G).$$

$\mathscr{C}$ will be referred to as the *basic* cosine function on G.

A mapping $G \ni a \mapsto \nu_a \in M(G)$ will be called a *homomorphism* if

$$\nu_a * \nu_b = \nu_{a+b} \tag{2.1}$$

for any $a, b \in G$. A homomorphism $G \ni a \mapsto \nu_a \in M(G)$ will be said to be bounded if $\sup_{a \in G} \|\nu_a\| < +\infty$. A homomorphism $G \ni a \mapsto \nu_a \in M_{\mathrm{ae}}(G)$ will be termed a c-*homomorphism* (the "c" is for "cosine") if

$$\nu_a + \nu_{-a} = \delta_a + \delta_{-a} \tag{2.2}$$

for each $a \in G$. A locally compact Abelian group that admits a c-homomorphism will be called a c-*group*. A locally compact Abelian group admitting a bounded c-homomorphism will be called a bc-*group*.

In the forthcoming discussion of group representability of bounded cosine functions, c-groups and bc-groups will play a crucial role. Prior to presenting details, we state two fundamental theorems about the shape of c-groups and bc-groups. Much of the paper will be devoted to proving these results.

Theorem 2.1. *A locally compact Abelian group G is a* c-group *if and only if $G^{(2)}$ is either a countable torsion group or a group isomorphic with $\mathbb{Z} \times F$, where F is a finite Abelian group.*

Theorem 2.2. *A locally compact Abelian group G is a* bc-group *if and only if $G^{(2)}$ is finite.*

Note that, by virtue of Theorems 2.1 and 2.2, the class of c-groups is essentially larger than that of bc-groups. In fact, the group $\mathbb{Z}$ is in the former class but not in the latter.

It is clear from the definition of a c-homomorphism that a given locally compact Abelian group G is a c-group if and only if the basic cosine function on G has a group representation. Similarly, a locally compact Abelian group G is a bc-group if and only if the basic cosine function on G has a bounded group representation. As we shall see shortly, these statements are particular cases of much stronger results (cf. Theorems 2.3, 2.4, and 2.5).

We start with a technicality.

Proposition 2.1. *Let G be a locally compact Abelian group such that $G^{(2)}$ is countable, and let A be a semitopological algebra with identity. Then every group representation of a A-valued continuous cosine function on G is regular.*

Proof. Since $G/G_{(2)} \simeq G^{(2)}$, it follows that $G/G_{(2)}$ is countable, and so, by Baire's theorem for locally compact regular spaces (cf. [13], § 5.28), $G/G_{(2)}$ is discrete. Thus $G_{(2)}$ is an open subgroup of G. Let $\mathscr{C} : G \to A$ be a continuous cosine function, and let $\mathscr{G} : G \to A$ be a homomorphism satisfying (1.2). If $a \in G_{(2)}$, then $a = -a$, and so $\mathscr{G}(a) = \mathscr{C}(a)$. Hence, since $\mathscr{C}$ is continuous, $\mathscr{G}$ is continuous on $G_{(2)}$. Now that $G_{(2)}$ is an open neighbourhood of 0 in G and the multiplication operation of A is separately continuous, it is easy to see that $\mathscr{G}$ is continuous on the whole of G. □

We have the following fundamental result:

Theorem 2.3. *Let A be a sequentially complete semitopological algebra with identity. If a locally compact Abelian group G is a* c-group, *then every A-valued bounded continuous cosine function on G has a regular group representation. Similarly, if a locally compact Abelian group G is a* bc-group, *then every A-valued bounded continuous cosine function on G has a bounded regular group representation.*

Proof. Let G be a locally compact Abelian group admitting a c-homomorphism $G \ni a \mapsto v_a \in M_{ae}(G)$, and let $\mathscr{C} : G \to A$ be a bounded continuous cosine function. For each $a \in G$, define an element $\mathscr{G}(a)$ of A by setting

$$\mathcal{G}(a) = \int_G \mathscr{C}(g)d\nu_a(g).$$

Here, of course, the integral reduces to a series with (at most) countably many A-valued summands, and the series is convergent in view of the sequential completeness of A and the boundedness of $\mathscr{C}$ (we draw upon the standing assumption that the topology of A is determined by a family of pseudonorms and is Hausdorff). In view of (1.1) and the boundedness of $\mathscr{C}$, the function $G \times G \ni (g, h) \mapsto \mathscr{C}(g)\mathscr{C}(h) \in A$ is bounded. Hence, since the multiplication operation of A is separately continuous, it follows that

$$\mathcal{G}(a)\mathcal{G}(b) = \int_{G \times G} \mathscr{C}(g)\mathscr{C}(h)d(\nu_a \otimes \nu_b)(g, h)$$

for any $a, b \in G$. Now, by (2.1) and the evenness of the ν_a ($a \in G$),

$$\int_{G \times G} \mathscr{C}(g)\mathscr{C}(h)d(\nu_a \otimes \nu_b)(g, h) = \frac{1}{2}\int_{G \times G} (\mathscr{C}(g+h) + \mathscr{C}(g-h))d(\nu_a \otimes \nu_b)(g, h)$$

$$= \int_{G \times G} \mathscr{C}(g+h)d(\nu_a \otimes \nu_b)(g, h) = \int_G \mathscr{C}(g)d(\nu_a * \nu_b)(g) = \mathcal{G}(a+b).$$

Thus $\mathcal{G} : a \mapsto \mathcal{G}(a)$ is a homomorphism from G into A. It is clear that $\mathcal{G}$ is bounded if the c-homomorphism $a \mapsto \nu_a$ is bounded. In view of (2.2) and the evenness of $\mathscr{C}$, $\mathcal{G}$ satisfies (1.2). Finally, by Theorems 2.1 and 2.2 (which will be established later without recourse to the theorem being proved), $G^{(2)}$ is at most countable, and so, by Proposition 2.1, $\mathcal{G}$ is continuous. □

Given a locally compact Abelian group G, let $C_0(G)$ be the space of all complex continuous functions on G vanishing at infinity, and let $C_u(G)$ be the space of all complex bounded uniformly continuous functions on G. Let λ_G denote the Haar measure of G, let $L^\infty(G)$ be the space of all classes of complex essentially bounded λ_G-measurable functions on G, and let $L^1(G)$ be the space of all classes of complex functions which are λ_G-integrable on G. Let $C_{0e}(G)$, $C_{ue}(G)$, $L^\infty_e(G)$, and $L^1_e(G)$ be the spaces of all even elements of $C_0(G)$, $C_u(G)$, $L^\infty(G)$, and $L^1(G)$, respectively. For $a \in G$, let T_a denote the operator of translation by a defined for functions by $(T_a f)(b) = f(a+b)$ and for measures by $T_a\mu = \mu * \delta_{-a}$. It is obvious that each T_a defines a linear isometry of $C_0(G)$, $C_u(G)$, $L^\infty(G)$, $L^1(G)$, and $M(G)$. For $a \in G$, let

$$\mathscr{C}(a) = \frac{1}{2}(T_a + T_{-a}). \tag{2.3}$$

Clearly, each $\mathscr{C}(a)$ defines a linear operator of unit norm on $C_{0e}(G)$, $C_{ue}(G)$, $L^\infty_e(G)$, $L^1_e(G)$, and $M_e(G)$, and moreover $\mathscr{C} : a \mapsto \mathscr{C}(a)$ is a cosine function in each of the spaces $C_{0e}(G)$, $C_{ue}(G)$, $L^\infty_e(G)$, $L^1_e(G)$, and $M_e(G)$. $\mathscr{C}$ is continuous in $C_{0e}(G)$, $C_{ue}(G)$, and $L^1_e(G)$ if each of these spaces is equipped with the usual norm topology, and is also continuous in $L^\infty_e(G)$ and $M_e(G)$ if each of the latter spaces is given the standard $*$-weak topology.

We are now in a position to state the following converse to Theorem 2.3:

Theorem 2.4. *Let G be a locally compact Abelian group, and let $\mathscr{C}$ be the cosine function on G given by* (2.3). *Then G is a* c-*group* (bc-*group*) *if any of the following statements is valid*:

(i) $\mathscr{C}: G \to \mathscr{L}(M_e(G))$ *has a (bounded) group representation*;

(ii) $\mathscr{C}: G \to \mathscr{L}(L^1_e(G))$ *has a (bounded) group representation*;

(iii) $\mathscr{C}: G \to \mathscr{L}(L^\infty_e(G))$ *has a (bounded) group representation*;

(iv) $\mathscr{C}: G \to \mathscr{L}(C_{ue}(G))$ *has a (bounded) group representation*;

(v) $\mathscr{C}: G \to \mathscr{L}(C_{0e}(G))$ *has a (bounded) group representation.*

Proof. If we add to the above list the statement:

(vi) G is a c-group (bc-group),

then, to prove the theorem, it will suffice to establish the validity of the following chain of implications:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)}.$$

Below we establish successively each of the implications forming the above chain.

(i) $\Rightarrow$ (ii). Let $\mathscr{G}$ be a representation of G in $M_e(G)$ such that (1.2) holds for the $\mathscr{L}(M_e(G))$-valued cosine function $\mathscr{C}$. To prove that (ii) follows from (i), it suffices to show that $L^1_e(G)$, regarded as a subspace of $M_e(G)$, is an invariant subspace for all the $\mathscr{G}(a)$ $(a \in G)$. Fix $a \in G$ arbitrarily. Let $f \in L^1_e(G)$, and let V be an open symmetric neighbourhood of 0 in G. If $b \in V$, then

$$\text{(2.4)} \qquad \|\mathscr{C}(b)\mathscr{G}(a)f - \mathscr{G}(a)f\| = \|\mathscr{G}(a)(\mathscr{C}(b)f - f)\| \leqq \|\mathscr{G}(a)\| \, \|\mathscr{C}(b)f - f\|.$$

Hereafter, adhering to the standard convention, we shall identify any given element k of $L^1(G)$ with the measure that is absolutely continuous with respect to λ_G and has k for density. Now, for each $g \in L^1(G)$ and each $h \in L^1_e(G)$, we have

$$g * h = \int_G h(b)\mathscr{C}(b) g \, d\lambda_G(b),$$

where the right-hand side is to be interpreted as the Bochner integral of the function $G \ni b \mapsto h(b)\mathscr{C}(b)g \in L^1(G)$. In view of (2.4), for any non-negative λ_G-measurable even function φ on G with support in V such that $\int_G \varphi \, d\lambda_G = 1$, we have

$$\text{(2.5)} \qquad \|(\mathscr{G}(a)f) * \varphi - \mathscr{G}(a)f\| = \Big\| \int_G \varphi(b)\big(\mathscr{C}(b)\mathscr{G}(a)f - \mathscr{G}(a)f\big) \, d\lambda_G(b) \Big\|$$
$$\leqq \|\mathscr{G}(a)\| \sup_{b \in V} \|\mathscr{C}(b)f - f\|.$$

As $L^1_e(G)$ is an ideal in $M_e(G)$, $(\mathscr{G}(a)f) * \varphi$ is an element of $L^1_e(G)$. Clearly, $\sup_{b \in V} \|\mathscr{C}(b)f - f\|$ tends to zero as V runs over a fundamental system of open symmetric neighbourhoods of 0 in G. Hence, in view of (2.5), $\mathscr{G}(a)f$ is the limit of a net in $L^1_e(G)$ (with values of the form $(\mathscr{G}(a)f) * \varphi)$. Since $L^1_e(G)$ is closed in $M_e(G)$, it follows that $\mathscr{G}(a)f$ belongs to $L^1_e(G)$. Thus $L^1_e(G)$ is an invariant subspace for all the $\mathscr{G}(a)$ $(a \in G)$.

(ii) $\Rightarrow$ (iii). Let $\mathscr{G}$ be a representation of G in $L^1_e(G)$ such that (1.2) holds for the $\mathscr{L}(L^1_e(G))$-valued cosine function $\mathscr{C}$. For a bounded linear operator T on $L^1_e(G)$, let T' denote the dual of T acting on $L^\infty_e(G)$. Clearly, $\mathscr{C}': a \mapsto \mathscr{C}(a)'$ is a cosine function in $L^\infty_e(G)$, $\mathscr{G}': a \mapsto \mathscr{G}(a)'$ is a representation of G in $L^\infty_e(G)$ such that (1.2) holds with $\mathscr{C}'$ replacing $\mathscr{C}$, and $\mathscr{G}'$ is bounded if and only $\mathscr{G}$ is bounded. Since $\mathscr{C}'$ coincides with the $\mathscr{L}(L^\infty_e(G))$-valued cosine function $\mathscr{C}$, we see that the latter cosine function has a group representation which may be assumed bounded if the $\mathscr{L}(L^1_e(G))$-valued cosine function $\mathscr{C}$ has a bounded group representation.

(iii) $\Rightarrow$ (iv). Let $\mathscr{G}$ be a representation of G in $L^\infty_e(G)$ such that (1.2) holds for the $\mathscr{L}(L^\infty_e(G))$-valued cosine function $\mathscr{C}$. The implication (iii) $\Rightarrow$ (iv) will follow once we show that $C_{ue}(G)$, regarded as a subspace of $L^\infty_e(G)$, is an invariant subspace for all the $\mathscr{G}(a)$ $(a \in G)$. To this end, we employ an argument analogous to the one used in establishing the implication (i) $\Rightarrow$ (ii). We suitably change spaces and norms, and choose φ to be continuous so as to ensure that, for $a \in G$ and $f \in C_{ue}(G)$, $(\mathscr{G}(a)f) * \varphi$ is in $C_{ue}(G)$.

(iv) $\Rightarrow$ (v). Let $\mathscr{G}$ be a representation of G in $C_{ue}(G)$ such that (1.2) holds for the $\mathscr{L}(C_{ue}(G))$-valued cosine function $\mathscr{C}$. To prove that (iv) follows from (v), it suffices to show that $C_{0e}(G)$, regarded as a subspace of $C_{ue}(G)$, is an invariant subspace for all the $\mathscr{G}(a)$ $(a \in G)$. For each $a \in G$, $C_{0e}(G) \ni f \mapsto (\mathscr{G}(a)f)(0) \in \mathbb{C}$ is a linear bounded functional on $C_{0e}(G)$. By the Riesz representation theorem, there exists a unique $v_a \in M_e(G)$ such that

$$(\mathscr{G}(a)f)(0) = \int_G f dv_a \quad (f \in C_{0e}(G)).$$

Clearly $\|v_a\| \leqq \|\mathscr{G}(a)\|$. Recall that for $f \in C_0(G)$ and $\mu \in M(G)$ the convolution $f * \mu$ is the element of $C_0(G)$ defined by

$$(f * \mu)(a) = \int_G f(a-b)\, d\mu(b) \quad (a \in G).$$

Now, if $f \in C_{0e}(G)$ and $a, b \in G$, then, since $\mathscr{G}(a)f$ and v_a are even, we have

$$(\mathscr{G}(a)f)(b) = (\mathscr{C}(b)\mathscr{G}(a)f)(0) = (\mathscr{G}(a)\mathscr{C}(b)f)(0) = (f * v_a)(b).$$

Thus, for each $a \in G$, the mapping $f \mapsto f * v_a$ transforms $C_{0e}(G)$ into itself, and so $C_{0e}(G)$ is an invariant subspace for all the $\mathscr{G}(a)$ $(a \in G)$.

(v) $\Rightarrow$ (vi). Let $\mathscr{G}$ be a representation of G in $C_{0e}(G)$ such that (1.2) holds for the $\mathscr{L}(C_{0e}(G))$-valued cosine function $\mathscr{C}$. By the argument from the previous paragraph, for each $a \in G$, there exists a unique $v_a \in M_e(G)$ such that

$$\mathscr{G}(a)f = f * v_a \quad (f \in C_{0e}(G))$$

and $\|v_a\| = \|\mathscr{G}(a)\|$. Hence, if $a, b \in C_{0e}(G)$, then

$$f * v_{a+b} = \mathscr{G}(a+b)f = \mathscr{G}(a)\mathscr{G}(b)f = f * v_b * v_a.$$

We thus see that (2.1) holds. Analogously, (1.2) gets translated into (2.2). Let π_a denote the mapping from $M(G)$ onto $M_a(G)$ that takes measures into their corresponding atomic parts. As is known, π_a is a contractive projection, is a homomorphism of convolution algebras, and maps $M_e(G)$ onto $M_{ae}(G)$. A moment's reflection shows that $G \ni a \mapsto \pi_a(v_a) \in M_{ae}(G)$ is a c-homomorphism such that $\|\pi_a(v_a)\| \leqq \|\mathscr{G}(a)\|$ for each $a \in G$. Consequently, G is a c-group, and when $\mathscr{G}$ is bounded, G is in fact a bc-group. □

As an immediate consequence of Theorems 2.3 and 2.4, we obtain the following:

Theorem 2.5. *Let G be a locally compact Abelian group. Consider the following conditions*:

(i) *for every sequentially complete semitopological algebra A with identity, any A-valued bounded continuous cosine function on G has a regular group representation*;

(i′) *for every Banach space E, any $\mathscr{L}(E)$-valued bounded* (*strongly*) *continuous cosine function on G has a regular group representation*;

(ii) *for every sequentially complete semitopological algebra A with identity, any A-valued bounded continuous cosine function on G has a bounded regular group representation*;

(ii′) *for every Banach space E, any $\mathscr{L}(E)$-valued bounded* (*strongly*) *continuous cosine function on G has a bounded regular group representation.*

Then conditions (i) *and* (i′) *are equivalent and are satisfied if and only if G is a* c-*group*; *similarly, conditions* (ii) *and* (ii′) *are equivalent and are satisfied if and only if G is a* bc-*group.*

3. Characterising c-groups and bc-groups: sufficient conditions

The purpose of this section is to prove the sufficiency parts of Theorems 2.1 and 2.2.

Let G be a locally compact Abelian group. We denote by $\hat{G}$ the dual group of G. The pairing between elements of G and $\hat{G}$ will be indicated by $(\cdot,\cdot)$. Given a measure $\mu \in M(G)$, we denote by $\hat{\mu}$ the Fourier-Stieltjes transform of μ, that is,

$$\hat{\mu}(\chi) = \int_G (a, -\chi)\, d\mu(a) \quad (\chi \in \hat{G}).$$

Let $\Phi(\hat{G})$ be the set of all odd functions φ on $\hat{G}$ having the following properties:

(i) $\varphi(\hat{G} \setminus (\hat{G})_{(2)}) \subset \{-1, 1\}$ and $\varphi((\hat{G})_{(2)}) = \{0\}$;

(ii) for each $a \in G$, there exists $\mu_a \in M_a(G)$ such that

$$\hat{\mu}_a(\chi) = \varphi(\chi)(1 - (2a, -\chi)) \quad (\chi \in \hat{G}). \tag{3.1}$$

Note that if $\varphi \in \Phi(\hat{G})$ and $\hat{G} \neq (\hat{G})_{(2)}$, then $\varphi(\hat{G} \setminus (\hat{G})_{(2)}) = \{-1, 1\}$.

Proposition 3.1. *Let G be a locally compact Abelian group such that $\Phi(\hat{G})$ is non-void, and let φ be a function in $\Phi(\hat{G})$. Retaining the notation from the definition above, for each $a \in G$, put*

$$\nu_a = \frac{1}{2}(\delta_a + \delta_{-a} + \mu_a * \delta_{-a}). \tag{3.2}$$

Then the mapping $G \ni a \mapsto \nu_a \in M_{\mathrm{ae}}(G)$ is a c*-homomorphism. If there exists $\mu \in M(G)$ such that $\hat{\mu} = \varphi$, then that* c*-homomorphism is bounded.*

Proof. By (3.1) and (3.2), for each $a \in G$ and each $\chi \in \hat{G}$,

$$\hat{\nu}_a(\chi) = \frac{1+\varphi(\chi)}{2}(a, \chi) + \frac{1-\varphi(\chi)}{2}(a, -\chi).$$

Taking into account that φ is an odd function mapping $\hat{G} \setminus (\hat{G})_{(2)}$ into $\{-1, 1\}$ and $(\hat{G})_{(2)}$ onto $\{0\}$, we see that, for each $a \in G$ and each $\chi \in \hat{G}$,

$$\hat{\nu}_a(\chi) = (a, \varepsilon(\chi)\chi),$$

where

$$\varepsilon(\chi) = \begin{cases} \varphi(\chi) & \text{if } \chi \in \hat{G} \setminus (\hat{G})_{(2)}, \\ 1 & \text{if } \chi \in (\hat{G})_{(2)}. \end{cases}$$

From this it follows that, for each $a \in G$, ν_a is even and the mapping $a \mapsto \nu_a$ is a c-homomorphism of G.

If $\varphi = \hat{\mu}$ for some $\mu \in M_{\mathrm{a}}(G)$, then, by (3.1), for each $a \in G$ we have $\mu_a = \mu - \mu * \delta_{2a}$, whence $\|\mu_a\| \leqq 2\|\mu\|$. Thus, on account of (3.2), the c-homomorphism $a \mapsto \nu_a$ is bounded. The result follows. □

Proposition 3.2. *The set $\Phi(\hat{\mathbb{Z}})$ is non-empty.*

Proof. Denoting by $\mathbb{T}$ the multiplicative group of complex numbers with unit modulus, let φ be the function on $\mathbb{T}$ given by

$$\varphi(e^{i\theta}) = \begin{cases} 1 & \text{if } 0 < \theta < \pi, \\ 0 & \text{if either } \theta = 0 \text{ or } \theta = \pi, \\ -1 & \text{if } \pi < \theta < 2\pi. \end{cases}$$

It is readily seen that φ has the Fourier expansion of the form

$$\varphi(e^{i\theta}) \sim \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\theta}{2k+1} = \frac{2}{\pi i} \sum_{k \in \mathbb{Z}} \frac{1}{2k+1} e^{i(2k+1)\theta}.$$

Hence, if $a \in \mathbb{Z}$, then

$$\begin{aligned}\varphi(e^{i\theta})(1-e^{-2ia\theta}) &\sim \frac{2}{i\pi}\left(\sum_{k\in\mathbb{Z}}\frac{1}{2k+1}e^{i(2k+1)\theta}-\sum_{k\in\mathbb{Z}}\frac{1}{2k+1}e^{i(2(k-a)+1)\theta}\right)\\ &=\frac{2}{i\pi}\sum_{k\in\mathbb{Z}}\left(\frac{1}{2k+1}-\frac{1}{2(k+a)+1}\right)e^{i(2k+1)\theta}\\ &=\frac{4a}{i\pi}\sum_{k\in\mathbb{Z}}\frac{1}{(2k+1)(2(k+a)+1)}e^{i(2k+1)\theta}.\end{aligned}$$

For each $a\in\mathbb{Z}$, let μ_a be the element of the space $l^1(\mathbb{Z})$ of all complex summable sequences on $\mathbb{Z}$ defined by

$$\mu_a(n)=\begin{cases}0 & \text{if } n\equiv 0 \pmod 2,\\ \dfrac{4a}{i\pi n(n+2a)} & \text{if } n\equiv 1 \pmod 2.\end{cases}\tag{3.3}$$

Of course, $l^1(\mathbb{Z})$ can be identified with $M_a(\mathbb{Z})$, and so each μ_a ($a\in\mathbb{Z}$) can be viewed as an atomic measure on $\mathbb{Z}$. Identifying in a standard way $\hat{\mathbb{Z}}$ with $\mathbb{T}$ and applying the Fourier inversion theorem, we obtain, for each $a\in\mathbb{Z}$ and each $0\leq\theta<2\pi$,

$$\hat{\mu}_a(e^{i\theta})=\varphi(e^{i\theta})(1-e^{-2ia\theta}).$$

Hence $\varphi\in\Phi(\hat{\mathbb{Z}})$. □

If G is a locally compact Abelian group, then by G_d we mean G with the discrete topology. By bG we denote the Bohr compactification of G.

Proposition 3.3. *Let G be a locally compact Abelian group. If $\Phi((G_d)^\wedge)$ is non-empty, then $\Phi(\hat{G})$ is also non-empty.*

Proof. Let $\varphi\in\Phi((G_d)^\wedge)$. Since $(G_d)^\wedge\cong b(\hat{G})$ and since $\hat{G}$ can canonically be embedded into $b(\hat{G})$ (as a Borel subgroup; cf. [11]), we may meaningfully speak about the restriction $\varphi_{|\hat{G}}$ of φ to $\hat{G}$. Clearly, $\varphi_{|\hat{G}}$ is an odd function mapping $\hat{G}\setminus(\hat{G})_{(2)}$ into $\{-1,1\}$ and $(\hat{G})_{(2)}$ onto $\{0\}$. Since $M_a(G_d)$ can be identified with $M_a(G)$, we see that, for each $a\in G$, there is $\mu_a\in M_a(G)$ such that (3.1) holds with φ replaced by $\varphi_{|\hat{G}}$. Hence $\varphi_{|\hat{G}}\in\Phi(\hat{G})$. □

For a subgroup H of a locally compact Abelian group G, we denote by $H^\perp$ the annihilator of H in $\hat{G}$, that is, the closed subgroup of $\hat{G}$ defined as

$$H^\perp=\{\chi\in\hat{G}:(a,\chi)=1 \text{ for all } a\in H\}.$$

Recall that if H is closed, then $(G/H)^\wedge\cong H^\perp$ and $\hat{H}\cong\hat{G}/H^\perp$. Note also that if H_1 and H_2 are closed subgroups of G such that $H_1\subset H_2$, then $(H_2/H_1)^\wedge\cong H_1^\perp/H_2^\perp$. Indeed, $G/H_2\cong(G/H_1)/(H_2/H_1)$, and so $H_2^\perp$ is topologically isomorphic with the annihilator of H_2/H_1 in $(G/H_1)^\wedge$, or equivalently with the annihilator of H_2/H_1 in $H_1^\perp$, which immediately implies the desired relation.

Proposition 3.4. *Let G be a locally compact Abelian group. Suppose that G contains a closed subgroup H satisfying $H^{(2)}=G^{(2)}$ and such that $\Phi(\hat{H})$ is non-empty. Then $\Phi(\hat{G})$ is non-empty.*

Proof. Let $\psi \in \Phi(\hat{H})$. Since $\hat{G}/H^\perp \cong \hat{H}$, we may assume that ψ is a function on $\hat{G}/H^\perp$. Let π be the canonical homomorphism from $\hat{G}$ onto $\hat{G}/H^\perp$. We have

$$\pi^{-1}\big((\hat{G}/H^\perp)_{(2)}\big) = (\hat{G})_{(2)}. \tag{3.4}$$

Indeed, a moment's reflection shows that an element χ of $\hat{G}$ belongs to $\pi^{-1}\big((\hat{G}/H^\perp)_{(2)}\big)$ if and only if $2\chi \in H^\perp$ or, equivalently, if $\chi \in (H^{(2)})^\perp$. Since $H^{(2)} = G^{(2)}$, the latter condition is satisfied if and only if $\chi \in (G^{(2)})^\perp$, which in turn is equivalent to $\chi \in (\hat{G})_{(2)}$.

Let $\varphi = \psi \circ \pi$. Taking into account (3.4), we find that φ is an odd function on $\hat{G}$ such that $\varphi\big(G \setminus (\hat{G})_{(2)}\big) \subset \{-1, 1\}$ and $\varphi\big((\hat{G})_{(2)}\big) = \{0\}$. Given $h \in H$, let η_h be a measure in $M_a(H)$ such that

$$\hat{\eta}_h(\gamma) = \psi(\gamma)\big(1 - (2h, -\gamma)\big)$$

for each $\gamma \in \hat{G}/H^\perp$, and let ν_h be the unique measure in $M_a(G)$, concentrated on H, whose restriction to H coincides with η_h. Then, for each $h \in H$ and each $\chi \in \hat{G}$,

$$\hat{\nu}_h(\chi) = \varphi(\chi)\big(1 - (2h, -\chi)\big) \tag{3.5}$$

(cf. [13], § 31.46). Now, given $a \in G$, choose $h \in H$ so that $2a = 2h$ and set $\mu_a = \nu_h$. Using (3.5), one verifies at once that (3.1) holds for each $a \in G$. Consequently, φ is a member of $\Phi(\hat{G})$. □

Proposition 3.5. *Let G be a locally compact Abelian group such that $\Phi(\hat{G})$ is non-empty, and let K be a locally compact Abelian group such that $K^{(2)}$ is a singleton. Then $\Phi\big((G \times K)^\wedge\big)$ is non-empty.*

Proof. The result follows immediately from the foregoing proposition upon taking $G \times \{0\}$ for H. □

For any function f on a group G, we denote by If the function on G given by

$$If(a) = f(-a) \quad (a \in G),$$

and, for any Borel measure μ on G, we designate by $I\mu$ the Borel measure on G given by

$$I\mu(E) = \mu(-E) \quad (E \in \mathfrak{B}(G)).$$

Proposition 3.6. *Let G be a countable torsion Abelian group with the discrete topology. Then $\Phi(\hat{G})$ is non-empty.*

Proof. Let $\{F_n\}_{n \in \mathbb{N}}$ be a collection of finite subgroups of G such that $F_n \subset F_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} F_n = G$. Then $\{F_n^\perp\}_{n \in \mathbb{N}}$ is a collection of closed subgroups of $\hat{G}$ such that $F_n^\perp \supset F_{n+1}^\perp$ for each $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} F_n^\perp = \{0\}$. Given $n \in \mathbb{N}$, $\hat{F}_n$ is topologically isomorphic with $\hat{G}/F_n^\perp$ and has the same cardinality as F_n. It follows that $F_n^\perp$ has finite index in $\hat{G}$, and is therefore an open subgroup of $\hat{G}$. Taking into account that, for each $\chi \in \hat{G}$, χ and

$-\chi$ belong to the same coset of $F_1^{\perp}$ if and only if $\chi \in m_2^{-1}(F_1^{\perp})$, one can easily define a function $f_1 : \hat{G} \to \{-1, 0, 1\}$ such that:

(i) $f_1(\hat{G} \setminus m_2^{-1}(F_1^{\perp})) \subset \{-1, 1\}$ and $f_1(m_2^{-1}(F_1^{\perp})) = \{0\}$;

(ii) f_1 is constant on the cosets of $F_1^{\perp}$ in $\hat{G}$;

(iii) f_1 is odd.

Since $F_2^{\perp}$ is a subgroup of $F_1^{\perp}$ of finite index (for $F_1^{\perp}/F_2^{\perp} \cong (F_2/F_1)^{\wedge}$), one can now define a function $f_2 : \hat{G} \to \{-1, 0, 1\}$ such that:

(i) $f_2(\hat{G} \setminus m_2^{-1}(F_2^{\perp})) \subset \{-1, 1\}$ and $f_2(m_2^{-1}(F_2^{\perp})) = \{0\}$;

(ii) f_2 is constant on the cosets of $F_2^{\perp}$ in $\hat{G}$;

(iii) f_2 is odd;

(iv) $f_2 = f_1$ on $\hat{G} \setminus m_2^{-1}(F_1^{\perp})$.

Continuing the process, we obtain a sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions from $\hat{G}$ into $\{-1, 0, 1\}$ such that for each $n \in \mathbb{N}$:

(i) $f_n(\hat{G} \setminus m_2^{-1}(F_n^{\perp})) \subset \{-1, 1\}$ and $f_n(m_2^{-1}(F_n^{\perp})) = \{0\}$;

(ii) f_n is constant on the cosets of $F_n^{\perp}$ in $\hat{G}$;

(iii) f_n is odd;

(iv) $f_{n+1} = f_n$ on $\hat{G} \setminus m_2^{-1}(F_n^{\perp})$.

For each $n \in \mathbb{N}$, f_n can be written as $f_n = g_n \circ \pi_n$, where π_n is the canonical homomorphism from $\hat{G}$ onto $\hat{G}/F_n^{\perp}$, and g_n is a unique function on $\hat{G}/F_n^{\perp}$. Since $\hat{G}/F_n^{\perp} \cong \hat{F}_n$ and since F_n is finite, $\hat{g}_n$ can be identified with an element of $M_a(F_n)$. Let ν_n be the unique measure in $M_a(G)$, concentrated on F_n, such that the restriction of ν_n to F_n coincides with $I\hat{g}_n$. Applying the Fourier inversion theorem, it is easy to see that $f_n = \hat{\nu}_n$.

Let

$$V = \bigcup_{n \in \mathbb{N}} f_n^{-1}(\{1\}).$$

As all the f_n ($n \in \mathbb{N}$) are odd, we have

$$-V = \bigcup_{n \in \mathbb{N}} f_n^{-1}(\{-1\}),$$

and so $V \cap (-V) = \emptyset$. Moreover

$$\hat{G} \setminus (V \cup (-V)) = \bigcap_{n \in \mathbb{N}} f_n^{-1}(\{0\}) = \bigcap_{n \in \mathbb{N}} m_2^{-1}(F_n^{\perp}) = m_2^{-1}\Big(\bigcap_{n \in \mathbb{N}} F_n^{\perp}\Big) = (\hat{G})_{(2)},$$

showing that $V \cup (-V) = \hat{G} \setminus (\hat{G})_{(2)}$. With this information about V, we can now define a function φ on $\hat{G}$ as follows:

$$\varphi(\chi) = \begin{cases} 1 & \text{if } \chi \in V, \\ 0 & \text{if } \chi \in (\hat{G})_{(2)}, \\ -1 & \text{if } \chi \in -V. \end{cases}$$

It is clear that, for each $n \in \mathbb{N}$, $\varphi = f_n$ on $\hat{G} \setminus m_2^{-1}(F_n^\perp)$.

If $a \in G$, then $a \in F_n$ for some $n \in \mathbb{N}$. We shall verify that (3.1) holds if we take $\nu_n - \nu_n * \delta_{2a}$ for μ_a, and thereby shall show that $\varphi \in \Phi(\hat{G})$. To this end, note that

$$\hat{\mu}_a(\chi) = f_n(\chi)\big(1 - (2a, -\chi)\big) \tag{3.6}$$

for each $\chi \in \hat{G}$. Now, if $\chi \in \hat{G} \setminus m_2^{-1}(F_n^\perp)$, then $\varphi(\chi) = f_n(\chi)$; this together with (3.6) yields (3.1) for $\chi \in \hat{G} \setminus m_2^{-1}(F_n^\perp)$. On the other hand, if $\chi \in m_2^{-1}(F_n^\perp)$, then $(2a, -\chi) = (-a, 2\chi) = 1$, and hence

$$f_n(\chi)\big(1 - (2a, -\chi)\big) = \varphi(\chi)\big(1 - (2a, -\chi)\big) = 0\,;$$

this, by (3.6), implies (3.1) for $\chi \in \hat{G} \setminus m_2^{-1}(F_n^\perp)$. The result follows. □

Proposition 3.7. *Let G be a locally compact Abelian group such that $G^{(2)}$ is a countable torsion group. Then $\Phi(\hat{G})$ is non-empty.*

Proof. By virtue of Proposition 3.3, we may assume (without loss of generality) that the topology of G is discrete. Since $G/G_{(2)} \simeq G^{(2)}$, it follows that $G/G_{(2)}$ is countable. Let S be a complete set of representatives modulo $G_{(2)}$, and let H be the smallest subgroup of G containing S. Clearly, S has the same cardinality as $G/G_{(2)}$, so it is countable. Accordingly, H is countable too. It is evident that $H^{(2)} = G^{(2)}$. Since $G^{(2)}$ is a torsion group, so too is H. Now, by Proposition 3.6, $\Phi(\hat{H})$ is non-empty and, by Proposition 3.4, also $\Phi(\hat{G})$ is non-empty. □

Proposition 3.8. *Let G be a locally compact Abelian group such that $\Phi(\hat{G})$ is non-empty, and let F be a finite Abelian group. Then $\Phi\big((G \times F)^\wedge\big)$ is non-empty.*

Proof. Let $\varphi_1 \in \Phi(\hat{G})$, let φ_2 be the characteristic function of $(\hat{F})_{(2)}$ defined on $\hat{F}$, and let φ_3 be an odd function on $\hat{F}$ mapping $\hat{F} \setminus (\hat{F})_{(2)}$ into $\{-1, 1\}$ and $(\hat{F})_{(2)}$ onto $\{0\}$. Let μ be the measure in $M_a(F)$, concentrated on $F^{(2)}$, whose restriction to $F^{(2)}$ coincides with the Haar measure of $F^{(2)}$. Since $(\hat{F})_{(2)} = (F^{(2)})^\perp$, it follows that

$$\varphi_2 = \hat{\mu}\,. \tag{3.7}$$

Moreover, for each $y \in F$ and each $\beta \in \hat{F}$,

$$\varphi_2(\beta) = \varphi_2(\beta)(2y, -\beta)\,. \tag{3.8}$$

Obviously, since F is finite, there exists $\varrho \in M_a(F)$ such that

$$\varphi_3 = \hat{\varrho}\,. \tag{3.9}$$

For each $\chi = (\alpha, \beta) \in \hat{G} \times \hat{F}$, let

$$\varphi(\chi) = \varphi_1(\alpha)\varphi_2(\beta) + \varphi_3(\beta)\,.$$

If $a = (x, y) \in G \times F$, then $2a = (2x, 2y)$ and, by (3.8),

$$\begin{aligned}\varphi(\chi)(1 - (2a. -\chi)) = {} & \varphi_1(\alpha)(1 - (2x, -\alpha))\varphi_2(\beta) \\ & + \varphi_3(\beta) - (2x, -\alpha)\varphi_3(\beta)(2y, -\beta)\,.\end{aligned} \tag{3.10}$$

For each $x \in G$, let μ_x be the measure in $M_a(G)$ such that

$$\hat{\mu}_x(\alpha) = \varphi_1(\alpha)(1 - (2x, -\alpha)) \quad (\alpha \in \hat{G})\,.$$

Comparing the latter equality with (3.7), (3.9) and (3.10), and identifying in a standard way $(G \times F)^\wedge$ with $\hat{G} \times \hat{F}$, we see that the function $\chi \mapsto \varphi(\chi)(1 - (2a, -\chi))$ is the Fourier transform of the measure ν_a given by

$$\nu_a = \mu_x \otimes \mu + \delta_0 \otimes \varrho - \delta_{2x} \otimes (\varrho * \delta_{2y})\,.$$

Now, if $\beta \notin (\hat{F})_{(2)}$, then $\varphi_2(\beta) = 0$ and hence $\varphi(\chi) = \varphi_3(\beta)$. Similarly, if $\beta \in (\hat{F})_{(2)}$, then $\varphi_2(\beta) = 1$ and $\varphi_3(\beta) = 0$, and so $\varphi(\chi) = \varphi_1(\alpha)$. These observations combined with the fact that $\chi \in (\hat{G} \times \hat{F})_{(2)}$ if and only if $\alpha \in (\hat{G})_{(2)}$ and $\beta \in \hat{F}_{(2)}$ show that φ is an odd function mapping $\hat{G} \times \hat{F} \setminus (\hat{G} \times \hat{F})_{(2)}$ into $\{-1, 1\}$ and $(\hat{G} \times \hat{F})_{(2)}$ onto $\{0\}$. Consequently, φ is an element of $\Phi((G \times F)^\wedge)$. □

For each $n \in \mathbb{N}$, we denote by $\mathbb{Z}(n)$ the cyclic group of order n.

Proposition 3.9. *Let G be an Abelian group such that $G^{(2)}$ is finite. Then $G \simeq \mathbb{Z}(2)^{\mathfrak{m}*} \times F$, where $\mathfrak{m}$ is a cardinal number and F is a finite Abelian group.*

Proof. Clearly, G is of bounded order. Hence, by a theorem of Prüfer-Baer [17], [1] (see also [13], § A.25, and [10], Theorem 17.2), G is isomorphic with $\prod^*_{i \in I} \mathbb{Z}(p_i^{r_i})$, where only finitely many distinct primes p_i and positive integers r_i occur. Since $G^{(2)}$ is finite, among the factors $\mathbb{Z}(p_i^{r_i})$ there are only finitely many ones with $p_i = 2$ and $r_i > 1$, and with $p_i > 2$. The proposition follows. □

Note that, for any Abelian group G, $G_{(2)}$ is isomorphic with $\mathbb{Z}(2)^{\mathfrak{m}*}$, where $\mathfrak{m}$ is a cardinal number. This follows immediately from $G_{(2)}$ being in a natural way a vector space over the field $\mathbb{Z}(2)$. One might for a moment think that perhaps Proposition 3.9 can be established by showing that $G_{(2)}$ is a direct product of $G_{(2)}$ and a group isomorphic with $G^{(2)}$. Taking $\mathbb{Z}(4)$ for G shows, however, that $G_{(2)}$ may fail to be a direct factor of G.

Theorem 3.1. *Let G be a locally compact Abelian group such that $G^{(2)}$ is either a countable torsion group or a group isomorphic with $\mathbb{Z} \times F$, where F is a finite Abelian group. Then $\Phi(\hat{G})$ is non-empty.*

Proof. In view of Proposition 3.3, we may assume that G is discrete. If $G^{(2)}$ is a countable torsion group, then the theorem immediately follows upon applying Proposition 3.7. Suppose then that $G^{(2)} \simeq \mathbb{Z} \times F$, where F is a finite Abelian group. Let π be the canonical homomorphism from G onto $G/G_{(2)}$. Identifying $G^{(2)}$ with $G/G_{(2)}$ and F with a suitable subgroup of $G^{(2)}$, we find that $G/\pi^{-1}(F) \simeq \mathbb{Z}$. By a theorem on direct factorisation (cf. [13], § 25.30 (a)), $\pi^{-1}(F)$ is a direct factor of G, and so $G \simeq \mathbb{Z} \times \pi^{-1}(F)$. Clearly, $(\pi^{-1}(F))^{(2)}$ is isomorphic with F, and hence it is finite. By Proposition 3.9, $\pi^{-1}(F) \simeq \mathbb{Z}(2)^{\mathfrak{m}*} \times F'$, where $\mathfrak{m}$ is a cardinal number and F' is a finite Abelian group. Now the theorem follows upon applying Propositions 3.2, 3.5 and 3.8. □

As an immediate consequence of Proposition 3.1 and Theorem 3.1, we obtain the sufficiency part of Theorem 2.1 which for clarity we state below as a separate proposition.

Proposition 3.10. *Let G be a locally compact Abelian group G such that $G^{(2)}$ is either a countable torsion group or a group isomorphic with $\mathbb{Z} \times F$, where F is a finite Abelian group. Then G is a* c*-group.*

For a locally compact Abelian group G, we denote by $A_a(\hat{G})$ the space of all Fourier transforms of measures in $M_a(G)$.

Theorem 3.2. *Let G be a locally compact Abelian group such that $G^{(2)}$ is finite. Then $\Phi(G) \cap A_a(\hat{G})$ is non-empty.*

Proof. By Proposition 3.9, G is isomorphic with $\mathbb{Z}(2)^{\mathfrak{m}*} \times F$, where $\mathfrak{m}$ is a cardinal number and F is a finite Abelian group. Let ν be a measure in $M_a(F)$ such that $\hat{\nu}$ is an odd function mapping $\hat{F} \backslash (\hat{F})_{(2)}$ into $\{-1, 1\}$ and $(\hat{F})_{(2)}$ onto $\{0\}$. Set $\mu = \delta_0 \otimes \nu$. Then μ is an element of $M_a(G)$ and it is easy to verify that $\hat{\mu}$ belongs to $\Phi(\hat{G}) \cap A_a(\hat{G})$. □

As before, we see that Proposition 3.1 and Theorem 3.2 imply the sufficiency part of Theorem 2.2 which can be stated in the following form:

Proposition 3.11. *Let G be a locally compact Abelian group such that $G^{(2)}$ is finite. Then G is a* bc*-group.*

4. Decomposable groups

In this section, we single out a class of locally compact Abelian groups each member of which is decomposable in a certain sense, and characterise compact groups in this class. This characterisation will be of direct relevance in the subsequent section.

A locally compact Abelian group G will be called *decomposable* if there exists an open subset U of G such that $U \cup (-U) = G \backslash G_{(2)}$ and $U \cap (-U) = \emptyset$.

Proposition 4.1. *Any decomposable connected compact Abelian group different from a singleton is topologically isomorphic with $\mathbb{T}$.*

Proof. Let G be a decomposable connected compact Abelian group different from a singleton. By the connectedness of G, $\hat{G}$ is torsion free (cf. [13], § 24.25). Let $\{\chi_i\}_{i \in I}$ be a maximal indexed collection of independent elements of G, and let $\mathfrak{m}$ be the cardinality of the index set I. As is known, $\mathfrak{m}$ does not depend on the particular choice of the maximal family of independent elements of $\hat{G}$ and defines the so-called torsion-free rank of $\hat{G}$. By the maximality of $\{\chi_i\}_{i\in I}$, for each $\chi \in \hat{G}$, there exist integers $n(\chi)$ and $n_i(\chi)$ $(i \in I)$ such that $n_i(\chi) \neq 0$ for only finitely many $i \in I$, and $n(\chi)\chi = \sum_{i\in I} n_i(\chi)\chi_i$. By the independency of the χ_i $(i \in I)$, $n(\chi)$ can be taken to be non-zero so that – in particular – for each $i \in I$ the rational number $n_i(\chi)/n(\chi)$ makes sense; moreover, this number depends only on χ. One verifies at once that, for each $i \in I$, the function $\varrho_i : \chi \mapsto n_i(\chi)/n(\chi)$ is a homomorphism from $\hat{G}$ into the group $\mathbb{Q}$ of rational numbers. We claim that, for each $i \in I$, $\varrho_i(\hat{G})$ is isomorphic with $\mathbb{Z}$.

In fact, fixing $i \in I$ arbitrarily and, for a given homomorphism f, letting $\ker f$ denote the kernel of f, observe first that $(\varrho_i(\hat{G}))^\wedge \cong (\ker \varrho_i)^\perp$, where the annihilator is taken in $\hat{\hat{G}}$ identified with G. Being a subgroup of G, $(\ker \varrho_i)^\perp$ is decomposable. Let U be an open subset of $(\ker \varrho_i)^\perp$ such that $U \cup (-U) = (\ker \varrho_i)^\perp \setminus ((\ker \varrho_i)^\perp)_{(2)}$ and $U \cap (-U) = \emptyset$. Since $\varrho_i(\hat{G}) \subset \mathbb{Q}$, $\varrho_i(\hat{G})$ can be regarded as a subgroup of $\mathbb{R}$. Let α_i be the embedding of $\varrho_i(\hat{G})$ into $\mathbb{R}$, and let $\hat{\alpha}_i$ be the dual homomorphism from $\hat{\mathbb{R}}$ $(\cong \mathbb{R})$ onto a dense subgroup of $(\ker \varrho_i)^\perp$ given by

$$\big(r, \hat{\alpha}_i(t)\big) = e^{it\alpha_i(r)} \quad (r \in \varrho_i(\hat{G}),\ t \in \hat{\mathbb{R}}).$$

Let S be the closure of $\hat{\alpha}_i\big((0, +\infty)\big)$ in $(\ker \varrho_i)^\perp$. Clearly, S is a compact cancellative semigroup. By an elementary result from the theory of topological semigroups (cf. [13], § 9.16, and [2], Theorem 1.10), S is a group. Hence S contains $\hat{\alpha}_i(\hat{\mathbb{R}})$, and, since $\hat{\alpha}_i(\hat{\mathbb{R}})$ is dense in $(\ker \varrho_i)^\perp$, S coincides with $(\ker \varrho_i)^\perp$. Now, proceeding by *reductio ad absurdum*, suppose that $\varrho_i(\hat{G})$ is not isomorphic with $\mathbb{Z}$. Then $\varrho_i(\hat{G})$ is a dense subgroup of $\mathbb{R}$, and so $\hat{\alpha}_i$ is one-to-one. Consequently, $\hat{\alpha}_i^{-1}\big(((\ker \varrho_i)^\perp)_{(2)}\big) = \big(\hat{\alpha}_i^{-1}((\ker \varrho_i)^\perp)\big)_{(2)} = \{0\}$ so that $\hat{\alpha}_i^{-1}(U)$ and $\hat{\alpha}_i^{-1}(-U)$ are disjoint, open and closed subsets of $\mathbb{R} \setminus \{0\}$ whose union coincides with $\mathbb{R} \setminus \{0\}$. From this it follows that $\hat{\alpha}_i^{-1}(U)$ coincides either with $(0, +\infty)$ or with $(-\infty, 0)$. Substituting $-U$ for U if necessary, we may assume that $\hat{\alpha}_i^{-1}(U) = (0, +\infty)$. Then $\hat{\alpha}_i\big((0, +\infty)\big) \subset U$, and so, if we let $\bar{U}$ denote the closure of U in $(\ker \varrho_i)^\perp$, we see that $S \subset \bar{U}$, whence $\bar{U} = (\ker \varrho_i)^\perp$. On the other hand, it is clear that $-U \subset (\ker \varrho_i)^\perp \setminus \bar{U}$. This contradiction establishes the claim.

To complete the proof, it suffices to show that $\mathfrak{m} = 1$. Suppose the contrary. Let i_1 and i_2 be two distinct elements of I. Let $\varrho : \hat{G} \to \mathbb{Q}^2$ be the homomorphism defined by $\varrho = (\varrho_{i_1}, \varrho_{i_2})$. Then, in view of the fact that χ_{i_1} and χ_{i_2} are independent and both $\varrho_{i_1}(\hat{G})$ and $\varrho_{i_2}(\hat{G})$ are isomorphic with $\mathbb{Z}$, we see that $\varrho(\hat{G})$ is isomorphic with $\mathbb{Z}^2$. Since $\big(\varrho(\hat{G})\big)^\wedge \cong (\ker \varrho)^\perp$, it follows that $(\ker \varrho)^\perp \cong \mathbb{T}^2$. On the other hand, $(\ker \varrho)^\perp$ as a subgroup of G is decomposable. But clearly $\mathbb{T}^2$ is not decomposable, as $\mathbb{T}^2 \setminus (\mathbb{T}^2)_{(2)} = \mathbb{T}^2 \setminus \{(1,1), (-1,1), (1,-1), (-1,-1)\}$ is connected. This contradiction completes the proof. □

Proposition 4.2. *Any decomposable compact Abelian group that is not totally disconnected is topologically isomorphic with* $\mathbb{T} \times \mathbb{Z}(2)^{\mathfrak{m}} \times F$, *where* $\mathfrak{m}$ *is a cardinal number and* F *is a finite Abelian group.*

Proof. Let G be a decomposable compact Abelian group that is not totally disconnected. Let G_0 be the component of 0 in G. Since G is not totally disconnected, G_0 is not a singleton. Being a closed subgroup of G, G_0 is compact and decomposable. By Proposition

4.1, G_0 is topologically isomorphic with $\mathbb{T}$. Now it follows from a theorem on direct factorisation (cf. [13], § 25.31(a)) that $G \cong G_0 \times (G/G_0)$. The proof will be complete once we show that $G/G_0 \cong \mathbb{Z}(2)^{\mathfrak{m}} \times F$ for some cardinal number $\mathfrak{m}$ and some finite Abelian group F. We first show that $(G/G_0)^{(2)}$ is finite.

Suppose, on the contrary, that $(G/G_0)^{(2)}$ is infinite. Then, since

$$(G/G_0)^{(2)} \simeq (G/G_0)/((G/G_0)_{(2)}),$$

$(G/G_0)_{(2)}$ is a subgroup of G/G_0 of infinite index. Since G/G_0 is compact and $(G/G_0)_{(2)}$ is closed in G/G_0, it follows that $(G/G_0)_{(2)}$ is a subset of G/G_0 of null Haar measure. Consequently, there is a net $\{x_\alpha\}_{\alpha \in A}$ in $(G/G_0) \setminus (G/G_0)_{(2)}$ such that $\lim_A x_\alpha = 0$. Let U be an open subset of G such that $U \cup (-U) = G \setminus G_{(2)}$ and $U \cap (-U) = \emptyset$. Since, for each $\alpha \in A$, $G_0 + x_\alpha$ is connected and does not intersect $G_{(2)}$, it follows that either $G_0 + x_\alpha \subset U$ or $G_0 + x_\alpha \subset -U$. Substituting $-x_\alpha$ for x_α if necessary, we may assume that $G_0 + x_\alpha \subset U$ for each $\alpha \in A$. Since $x_\alpha \to 0$ as α tends to infinity along A, we see that G_0 is contained in the intersection of the closures in G of U and $-U$, and as such it is a subset of $G_{(2)}$. But the latter is incompatible with the fact that G_0 is isomorphic with $\mathbb{T}$. This contradiction establishes the finiteness of $(G/G_0)^{(2)}$.

If H is a compact group, then $H^{(2)}$ is a closed subgroup of H, and $(H^{(2)})^\wedge \cong \hat{H}/(H^{(2)})^\perp$. But $(H^{(2)})^\perp = (\hat{H})_{(2)}$, and $\hat{H}/(\hat{H})_{(2)} \simeq (\hat{H})^{(2)}$. Thus $(H^{(2)})^\wedge \simeq (\hat{H})^{(2)}$. Now, if $H^{(2)}$ is finite, then, since $H^{(2)}$ and $(H^{(2)})^\wedge$ have equal cardinalities, $(\hat{H})^{(2)}$ is also finite. Applying these observations to $H = G/G_0$ and using Proposition 3.9, we see that $(G/G_0)^\wedge \simeq \mathbb{Z}(2)^{\mathfrak{m}*} \times F$, where $\mathfrak{m}$ is a cardinal number and F is a finite Abelian group. By duality, it follows that G/G_0 is topologically isomorphic with $\mathbb{Z}(2)^{\mathfrak{m}} \times \hat{F}$. □

Proposition 4.3. *If G is a decomposable totally disconnected compact Abelian group, then $(\hat{G})^{(2)}$ is countable.*

Proof. Let G be a decomposable totally disconnected compact Abelian group. Proceeding by *reductio ad absurdum*, suppose that $(\hat{G})^{(2)}$ is uncountable. Let U be an open subset of G such that $U \cup (-U) = G \setminus G_{(2)}$ and $U \cap (-U) = \emptyset$. Since G is totally disconnected, it has a basis $\mathfrak{V}$ of neighbourhoods of 0 consisting of compact open subgroups of G (cf. [13], § 7.7). Choose $a_1 \in U$. Then $a_1 + V_1 \subset U$ for some $V_1 \in \mathfrak{V}$. Note that $V_1 \setminus G_{(2)} \neq \emptyset$. Indeed, otherwise V_1 would be contained in $G_{(2)}$ so that $G_{(2)}$ would be an open subgroup of G, and, since G is compact, $G_{(2)}$ would have finite index in G. Now, since $G/G_{(2)} \simeq G^{(2)}$, $G^{(2)}$ would be finite. But then, as we saw while establishing the foregoing proposition, $(\hat{G})^{(2)}$ would also be finite, a contradiction. Let $a_2 \in V_1 \setminus G_{(2)}$. Passing if necessary to $-a_2$, we may assume that $a_2 \in V_1 \cap U$. Now, since $V_1 \cap U$ is open, there exists $V_2 \in \mathfrak{V}$ such that $V_2 \subset V_1$ and $a_2 + V_2 \subset V_1 \cap U$. Continuing the process, we obtain a sequence $\{V_n\}_{n \in \mathbb{N}}$ in $\mathfrak{V}$ and a sequence $\{a_n\}_{n \in \mathbb{N}}$ in G such that, for each $n \in \mathbb{N}$,

$$V_{n+1} \subset V_n \tag{4.1}$$

and

$$a_{n+1} + V_{n+1} \subset V_n \cap U. \tag{4.2}$$

Since $0 \in V_n$, we have $G_{(2)} \subset V_n + G_{(2)}$ for each $n \in \mathbb{N}$, and so

$$G_{(2)} \subset \bigcap_{n \in \mathbb{N}} (V_n + G_{(2)}).$$

We claim that

$$\left(\bigcap_{n \in \mathbb{N}} (V_n + G_{(2)})\right) \setminus G_{(2)} \neq \emptyset.$$

Indeed, otherwise $\{V_n + G_{(2)}\}_{n \in \mathbb{N}}$ can serve as a basis of neighbourhoods of 0 in $G/G_{(2)}$, and so $G/G_{(2)}$ is metrisable. It is easy to see that $G/G_{(2)}$ is topologically isomorphic with $G^{(2)}$. In fact, since G is compact, $G/G_{(2)}$ is also compact. The mapping $m_2: G \to G^{(2)}$ is continuous, and so, by a fundamental property of the quotient topology, the induced isomorphism $m_2^*: G/G_{(2)} \to G^{(2)}$ is also continuous. Now, in view of the compactness of $G/G_{(2)}$, m_2^* is a homeomorphism. Having established the relation $G/G_{(2)} \cong G^{(2)}$, we now infer that $G^{(2)}$ itself is metrisable. Since $G^{(2)}$ is also compact, $(G^{(2)})^\wedge$ is countable. But, as we saw while proving the foregoing proposition, $(G^{(2)})^\wedge$ is isomorphic with $(\hat{G})^{(2)}$. Thus $(\hat{G})^{(2)}$ is countable. This contradiction establishes the claim.

Let $\mathscr{U}$ be a non-free ultrafilter on $\mathbb{N}$, and let $a_\infty = \lim_{n \to \mathscr{U}} a_n$ (cf. [4], §11.8; we resort to the notion of a limit with respect to an ultrafilter in order to be able to handle simultaneously and in a smooth way several accumulation points of sequences; the use of ultrafilters and respective limits is a mere technicality and can be avoided at the expence of a lengthening of the argument). In view of (4.1) and the fact that, for each $n \in \mathbb{N}$, V_n is closed and $a_{n+1} \in V_n$, it follows that

$$a_\infty \in \bigcap_{n \in \mathbb{N}} V_n. \tag{4.3}$$

We now show that, by modifying the sequence $\{a_n\}_{n \in \mathbb{N}}$ if necessary, we may always assume that $a_\infty \notin G_{(2)}$.

Suppose that initially $a_\infty \in G_{(2)}$. Let

$$b \in \left(\bigcap_{n \in \mathbb{N}} (V_n + G_{(2)})\right) \setminus G_{(2)}.$$

For each $n \in \mathbb{N}$, let $v_n \in V_n$ and $g_n \in G_{(2)}$ be such that $b = v_n + g_n$, and set $a_n' = a_n + v_n$. Since $v_{n+1} + V_{n+1} \subset V_{n+1}$, it follows from (4.2) that $a_{n+1}' + V_{n+1} \subset V_n \cap U$, and so $\{a_n'\}_{n \in \mathbb{N}}$ satisfies (4.2). Let $a_\infty' = \lim_{n \to \mathscr{U}} a_n'$ and $g_\infty = \lim_{n \to \mathscr{U}} g_n$. Clearly, $a_\infty' = a_\infty + b - g_\infty$. Since both a_∞ and g_∞ belong to $G_{(2)}$, we have $2a_\infty' = 2b$, and further, since $b \notin G_{(2)}$, we see that $a_\infty' \notin G_{(2)}$. Thus $\{a_n'\}_{n \in \mathbb{N}}$ is a desired modification.

Assuming now – as we may – that $a_\infty \notin G_{(2)}$, let $\tilde{U}$ be that of the sets U and $-U$ which contains a_∞. Let $W \in \mathfrak{B}$ be such that

$$a_\infty + W \subset \tilde{U}. \tag{4.4}$$

Choose $W' \in \mathfrak{B}$ so that $W' + W' \subset W$. Since a_∞ is a cluster point of $\{a_n\}_{n \in \mathbb{N}}$, there exist $k_1, k_2 \in \mathbb{N}$ such that $k_1 < k_2$ and $a_{k_i} - a_\infty \in W'$ $(i = 1,2)$. We see that both $a_{k_1} - a_{k_2}$ and $a_{k_2} - a_{k_1}$ belong to W, and, on account of (4.4), we have

$$a_\infty + a_{k_1} - a_{k_2} \in \tilde{U} \tag{4.5}$$

and

$$a_\infty + a_{k_2} - a_{k_1} \in \tilde{U},$$

the latter relation implying that

$$a_{k_1} - a_\infty - a_{k_2} \in -\tilde{U}. \tag{4.6}$$

On the other hand, by (4.1) and (4.2), $a_{k_2} \in V_{k_2-1} \subset V_{k_1}$, and, by (4.3), $a_\infty \in V_{k_1}$. Since V_{k_1} is a group, it follows that $a_\infty - a_{k_2} \in V_{k_1}$ and $-a_\infty - a_{k_2} \in V_{k_1}$. Now, in view of (4.2), we have

$$a_{k_1} + a_\infty - a_{k_2} \in U$$

and

$$a_{k_1} - a_\infty - a_{k_2} \in U.$$

But these last relations are incompatible with (4.5) and (4.6). This contradiction establishes the proposition. □

Now we are in a position to state the main conclusion of this section.

Theorem 4.1. *Let G be a compact Abelian group. Then G is decomposable if and only if either $(\hat{G})^{(2)}$ is a countable torsion group or G is topologically isomorphic with $\mathbb{T} \times \mathbb{Z}(2)^{\mathfrak{m}} \times F$, where $\mathfrak{m}$ is a cardinal number and F is a finite Abelian group.*

Proof. Necessity follows from Propositions 4.2 and 4.3.

To prove sufficiency, suppose that G is such that either $(\hat{G})^{(2)}$ is a countable torsion group or G is topologically isomorphic with $\mathbb{T} \times \mathbb{Z}(2)^{\mathfrak{m}} \times F$, where $\mathfrak{m}$ is a cardinal number and F is a finite Abelian group. Note that in the latter case $(\hat{G})^{(2)} \simeq \mathbb{Z} \times (\hat{F})^{(2)}$. By Theorem 3.1, in either case the set $\Phi(G)$ is non-empty. Let $\varphi \in \Phi(G)$ and $U = \varphi^{-1}(\{1\})$. Then, clearly, U is an open subset of G such that $U \cup (-U) = G \setminus G_{(2)}$ and $U \cap (-U) = \emptyset$. The result follows. □

5. Characterising c-groups and bc-groups: necessary conditions

The aim of this section is to prove the necessity parts of Theorem 2.1 and 2.2. We shall formulate these parts as separate propositions.

Proposition 5.1. *If a locally compact Abelian group G is a* c*-group, then $G^{(2)}$ is either a countable torsion group or a group isomorphic with $\mathbb{Z} \times F$, where F is a finite Abelian group.*

Proof. Let $G \ni a \mapsto v_a \in M_{\mathrm{ae}}(G)$ be a c-homomorphism of G. Identifying $M_{\mathrm{ae}}(G)$ with $M_{\mathrm{ae}}(G_d)$ and taking account of (2.1), we see that, for every $\chi \in (G_d)^\wedge \cong b(\hat{G})$, the function $a \mapsto \hat{v}_a(\chi)$ is a homomorphism from G_d into the multiplicative group $\mathbb{C} \setminus \{0\}$. Since, by (2.2), for each $a \in G$,

$$\hat{\nu}_a(\chi) + (\hat{\nu}_a(\chi))^{-1} = (a, \chi) + (a, -\chi) \tag{5.1}$$

and the right-hand side of this equality has modulus no greater than 2, it follows that the homomorphism $a \mapsto \hat{\nu}_a(\chi)$ takes on values in a bounded subgroup of $\mathbb{C} \setminus \{0\}$, and hence is a character of G_d. Moreover, (5.1) and the uniqueness of Fourier expansion ensure that, for each $\chi \in b(\hat{G})$, the character $a \mapsto \hat{\nu}_a$ takes the form

$$\hat{\nu}_a(\chi) = (a, \varepsilon(\chi)\chi) \quad (a \in G)$$

for some function ε on $b(\hat{G})$ with values in $\{-1, 1\}$. Observe that, in view of the last equality and evenness of the ν_a $(a \in G)$, the restriction of ε to $b(\hat{G}) \setminus (b(\hat{G}))_{(2)}$ is odd. Let $\varphi(\chi)$ be the function on $b(\hat{G})$ defined by

$$\varphi(\chi) = \begin{cases} \varepsilon(\chi) & \text{if } \chi \in b(\hat{G}) \setminus (b(\hat{G}))_{(2)}, \\ 0 & \text{if } \chi \in (b(\hat{G}))_{(2)}. \end{cases} \tag{5.2}$$

For each $a \in G$, let $\mu_a = 2\nu_a * \delta_a - \delta_{2a} - \delta_0$. Direct computation shows that

$$\hat{\mu}_a(\chi) = \varphi(\chi)(1 - (2a, -\chi)) \quad (a \in G,\ \chi \in b(\hat{G})). \tag{5.3}$$

If $\chi \in b(\hat{G}) \setminus (b(\hat{G}))_{(2)}$, then there exist $a \in G$ and an open neighbourhood V of χ such that $(2a, -\gamma) \neq 1$ for all $\gamma \in V$. From this and (5.3) it follows that the restriction of φ to $b(\hat{G}) \setminus (b(\hat{G}))_{(2)}$ is continuous. Let $U = \varphi^{-1}(\{1\})$. Since $b(\hat{G}) \setminus (b(\hat{G}))_{(2)}$ is an open subset of $b(\hat{G})$ and the restriction of φ to $b(\hat{G}) \setminus (b(\hat{G}))_{(2)}$ is odd and continuous, it follows that U is an open subset of $b(\hat{G})$ such that $U \cup (-U) = b(\hat{G}) \setminus (b(\hat{G}))_{(2)}$ and $U \cap (-U) = \emptyset$. By Theorem 4.1 and the fact that $(b(\hat{G}))^\wedge \cong G_d$, either $(G_d)^{(2)}$ is a countable torsion group or $b(\hat{G})$ is topologically isomorphic with $\mathbb{T} \times \mathbb{Z}(2)^{\mathfrak{m}} \times F$, where $\mathfrak{m}$ is a cardinal number and F is a finite Abelian group, in the latter case G_d being isomorphic with $\mathbb{Z} \times \mathbb{Z}(2)^{\mathfrak{m}*} \times \hat{F}$. Hence $G^{(2)}$ is either a countable torsion group or a group isomorphic with $\mathbb{Z} \times (\hat{F})^{(2)}$. □

Formulated in a different manner, our next result has already appeared in [3]. The proof given below is much simpler than the corresponding proof in [3] (namely, that accompanying Théorème 3.2).

Proposition 5.2. *If a locally compact Abelian group G is a* bc-*group, then $G^{(2)}$ is finite.*

Proof. Let G be a locally compact Abelian group admitting a bounded c-homomorphism $G \ni a \mapsto \nu_a \in M_{\mathrm{ae}}(G)$. Let $\mathrm{AP}(G)$ be the space of all complex almost periodic functions on G, and let $l^\infty(G)$ be the space of all complex bounded functions on G. Let m be a Banach mean on $l^\infty(G)$, that is, a bounded linear functional on $l^\infty(G)$ satisfying the following conditions:

(i) $\|m\| = 1 = m(1)$;

(ii) $m(T_a f) = m(f)$ for each $f \in l^\infty(G)$ and each $a \in G$.

The existence of such an invariant mean is ensured by a theorem of Day [5] (see also [13], § 17.5, and [12], Theorem 1.2.1). For each $a \in G$, set $\mu_a = 2\nu_a * \delta_a - \delta_{2a} - \delta_0$. Since, of course, $\sup_{a \in G} \|\mu_a\| < +\infty$, the mapping

$$\mathrm{AP}(G) \ni f \mapsto m_a\bigl(\int_G f d\mu_a\bigr) \in \mathbb{C}$$

is a well-defined functional on $\mathrm{AP}(G)$; here the subscript a in m_a indicates that the action of m refers to the dummy variable a. Since the space $\mathrm{AP}(G)$ can canonically be identified with the space of all complex continuous functions on bG, the above functional can be identified with a uniquely determined measure μ in $M(bG)$ such that

$$\hat{\mu}(\chi) = m_a\bigl(\hat{\mu}_a(\chi)\bigr) \tag{5.4}$$

for each $\chi \in (bG)^\wedge \cong (\hat{G})_d$. Repeating the argument from the proof to the preceding proposition, we see that there exists an odd function φ on $\hat{G}$ such that $\varphi(\hat{G} \setminus (\hat{G})_{(2)}) \subset \{-1, 1\}$, $\varphi((\hat{G})_{(2)}) = \{0\}$, and

$$\hat{\mu}_a(\chi) = \varphi(\chi)\bigl(1 - (2a, -\chi)\bigr) \quad (a \in G,\ \chi \in \hat{G}).$$

Comparing the last equality with (5.4), we see that

$$\hat{\mu}(\chi) = \varphi(\chi) m_a\bigl(1 - (2a, -\chi)\bigr)$$

for each $\chi \in \hat{G}$. Taking into account that

$$m_a\bigl((2a, -\chi)\bigr) = \begin{cases} 1 & \text{if } \chi \in (\hat{G})_{(2)}, \\ 0 & \text{if } \chi \in \hat{G} \setminus (\hat{G})_{(2)}, \end{cases}$$

we find that $\hat{\mu}(\chi) = \varphi(\chi)$ for each $\chi \in \hat{G}$. Accordingly, the measure μ is odd and

$$(\mu * \mu)^\wedge = (\hat{\mu})^2 = 1_{\hat{G} \setminus (\hat{G})_{(2)}}. \tag{5.5}$$

If we identify the Haar measure $\lambda_{(bG)^{(2)}}$ with a suitable measure in $M(bG)$ concentrated on $(bG)^{(2)}$, then clearly $\hat{\lambda}_{(bG)^{(2)}} = 1_{(\hat{G})_{(2)}}$. This identity together with (5.5) shows that

$$\mu * \mu = \delta_0 - \lambda_{(bG)^{(2)}}. \tag{5.6}$$

Now, proceeding by *reductio ad absurdum*, suppose that the group $G^{(2)}$ is infinite. Then the compact group $(bG)^{(2)}$ is also infinite and, correspondingly, the Haar measure $\lambda_{(bG)^{(2)}}$ is continuous. Passing to atomic parts in both sides of (5.6), we see that $\pi_a(\mu) * \pi_a(\mu) = \delta_0$. Thus $|(\pi_a(\mu))^\wedge(0)| = 1$. On the other hand, since μ is odd, so too is $\pi_a(\mu)$, and hence $(\pi_a(\mu))^\wedge(0) = 0$. This contradiction completes the proof. □

6. Concluding remarks

The aim of this final section is to present some results contributing to the theory of single operators in Banach spaces. These results will be consequences of the fact that $\mathbb{Z}$ is a c-group but not a bc-group.

For $n \in \mathbb{N}$, let $T_n(X)$ denote the Chebychev polynomial of degree n defined as

$$T_n(X) = \sum_{k=0}^{[n/2]} \binom{n}{2k} X^{n-2k}(X^2-1)^k,$$

where $[a]$ designates the integral part of the real number a. Set $T_0(X) = 1$ and $T_n(X) = T_{-n}(X)$ for each negative integer n. It is straightforward to verify that if A is an algebra with identity, then for each $a \in A$ the function $\mathbb{Z} \ni n \mapsto T_n(a) \in A$ is an A-valued cosine function on $\mathbb{Z}$, and conversely any cosine function $\mathscr{C} : \mathbb{Z} \to A$ assumes the form

$$\mathscr{C}(n) = T_n(a) \quad (n \in \mathbb{Z})$$

for some $a \in A$ (it suffices to take $\mathscr{C}(1)$ for a). By Theorems 2.1 and 2.3, we have the following:

Theorem 6.1. *Let A be a sequentially complete semitopological algebra with identity. If $a \in A$ is such that $\{T_n(a)\}_{n \in \mathbb{N}}$ is bounded, then there exists an invertible element b of A such that*

$$T_n(a) = \frac{1}{2}(b^n + b^{-n}) \tag{6.1}$$

for each $n \in \mathbb{N}$.

Inspection of the proofs of Theorems 2.1 and 2.3 (with special attention paid to (3.2) and (3.3)) reveals that (6.1) holds with b equal to

$$\frac{2i}{\pi} e + a - \frac{4i}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} T_{2k}(a),$$

where e denotes, of course, the identity of A.

It follows from Theorems 2.2 and 2.4 that there exist Banach spaces E, composed of even sequences indexed by $\mathbb{Z}$, and there exist elements a of $\mathscr{L}(E)$ satisfying $\|T_n(a)\| = 1$ for each $n \in \mathbb{N}$ such that if an invertible element b of $\mathscr{L}(E)$ satisfies (6.1), then $\sup_{n \in \mathbb{Z}} \|b^n\| = +\infty$. As, by virtue of Theorem 6.1, for any such a there is an invertible element b of $\mathscr{L}(E)$ satisfying (6.1), we see that the following holds true:

Theorem 6.2. *There exist Banach spaces E and invertible elements b of the corresponding algebras $\mathscr{L}(E)$ satisfying $\|b^n + b^{-n}\| = 2$ for each $n \in \mathbb{N}$ and having the following property: If c is an invertible element of $\mathscr{L}(E)$ such that*

$$b^n + b^{-n} = c^n + c^{-n} \tag{6.2}$$

for each $n \in \mathbb{N}$, then $\sup_{n \in \mathbb{Z}} \|c^n\| = +\infty$.

It should be stressed that none of the spaces E in the above theorem are isomorphic to a Hilbert space. Indeed, as mentioned in the Introduction, if H is a Hilbert space and

$b \in \mathscr{L}(H)$ is invertible and satisfies $\sup_{n \in \mathbb{N}} \|b^n + b^{-n}\| < +\infty$, then there exists an invertible element c of $\mathscr{L}(H)$ with $\sup_{n \in \mathbb{Z}} \|c^n\| < +\infty$ for which (6.2) holds. By a theorem of Wermer [19] (see also [6], Lemma XV.6.1), such c must then necessarily be similar to a unitary operator.

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