JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **165**, 192–215 (1992)

Circularly Symmetric Eikonal Equations and Non-uniqueness in Computer Vision

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Submitted by Avner Friedman

Received May 18, 1990

1. INTRODUCTION

The eikonal equation

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2=\mathscr{E}(x, y), \tag{1}$$

which arises naturally in wavefront analysis and in the development of special methods for integrating Hamilton's equations (the Jacobi–Hamilton method), has long attracted the attention of physicists and mathematicians. More recently, there has been a resurgence of interest in the eikonal equation as a result of its applicability in an area of computer vision. One of the issues considered in the latter context has been that of determining whether or not a particular eikonal equation exhibits many solutions defined over a given domain. In this paper, we shall offer insight into this issue by presenting a non-uniqueness result of significance for the foundations of computer vision.

A monochrome photograph of a smooth object will typically exhibit brightness variation, or *shading*. Of interest to researchers in computer vision is the problem of how object shape may be extracted from image shading. This *shape-from-shading problem* has been shown by Horn ([6];

0022-247X/92 $3.00

see also Horn and Brooks [7, pp. 123–172], where the same article appears in a collection of seminal papers in the field) to correspond to that of solving a first-order partial differential equation. Specifically, one seeks a function $u(x, y)$, representing surface depth in the direction of the z-axis, satisfying the *image irradiance equation*

$$R\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = E(x, y)$$

over Ω. Here, R is a known function (the so-called *reflectance map*) capturing the illumination and surface reflecting conditions, E is an image formed by (orthographic) projection of light along the z-axis onto a plane parallel to the xy-plane, and Ω is the image domain. In this formulation, it is implicitly assumed that

a small surface portion reflects light independently of its position in space. Thus, scene radiance emitted in a given direction is dependent only on the illumination, the light-scattering properties of the surface material, and the surface normal. By implication, light sources are infinitely far away, and internal surface reflections are disallowed.

image irradiance is equal to the projected scene radiance.

An interesting case obtains when the reflectance map is specified so as to correspond to the situation in which an overhead, distant point-source illuminates a *Lambertian surface*. A small portion of such a surface acts as a perfect diffuser appearing equally bright from all directions. At first, this might seem to imply that Lambertian surfaces cannot exhibit other than constant shading. However, a curved object will, in general, receive illumination that differs in strength across the surface due to surface foreshortening, and it is this that will be responsible for variation in image brightness. If a small surface portion with normal direction $(-\partial u/\partial x, -\partial u/\partial y, 1)$ is illuminated by a distant, overhead point-source of unit power in direction $(0, 0, 1)$, then, according to Lambert's law, the emitted radiance and, in view of the aforementioned assumptions, the reflectance map are given by the cosine of the angle between the two directions, namely $((\partial u/\partial x)^2 + (\partial u/\partial y)^2 + 1)^{-1/2}$. Thus, if $E(x, y)$ denotes the corresponding image, the image irradiance equation for the above situation takes the form

$$\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 1\right]^{-1/2} = E(x, y).$$

Noting that $0 < E(x, y) \leqslant 1$, we may safely let $\mathscr{E}(x, y) = (E(x, y))^{-2} - 1$ and rewrite the above equation as (1).

Given an image of some particular shape, the natural question arises as to whether it could also be the image of other shapes. For Lambertian surfaces illuminated by an overhead point-source, this reduces to the problem of finding all solutions of (1) over some domain. Note that if u is a solution of (1), then so too is any member of the family $\pm u + k$, where k is an arbitrary constant. Thus, the image of the surface S formed by the graph of u will be preserved under either a depth-shift of S along the z-axis, the inversion of S with respect to the xy-plane, or a combination of these transformations. These surfaces may clearly be said to possess a common shape. Of interest in computer vision is the situation of essential uniqueness in which a family of the type specified above constitutes, within some class of functions, the complete set of solutions to an equation of the form given in (1).

Uniqueness of this kind has been demonstrated for Eq. (1) in which

$$\mathscr{E}(x, y) = \frac{x^2 + y^2}{1 - x^2 - y^2}.$$

Deift and Sylvester [4] and, independently, Brooks [1] proved that $\pm(1 - x^2 - y^2)^{1/2} + k$ are the only C^2 solutions to this equation over the unit disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. All of these solutions are hemispherical in shape. Interestingly, this result fails in the class of C^1 solutions.

In an effort to obtain a more general result, Bruss ([3]; see also [7, pp. 69–88]), in perhaps the major work in the uniqueness area, asserted the following: if R is a positive number, $D(R)$ is the disc in the xy-plane with radius R centred at the origin, and f is a continuous function on $[0, R)$ of class C^2 over $(0, R)$ such that

(i) $f(0) = 0$ and $f(r) > 0$ for $0 < r < R$,

(ii) $\lim_{r \to 0} f'(r) = 0$, $\lim_{r \to 0} f''(r)$ exists and is positive,

(iii) $\lim_{r \to R} f(r) = +\infty$,

then all solutions of class C^2 to (1) in $D(R)$ with

$$\mathscr{E}(x, y) = f(\sqrt{x^2 + y^2}) \tag{2}$$

take the form

$$\pm \int_0^{\sqrt{x^2 + y^2}} \sqrt{f(\tau)}\, d\tau + k,$$

and so are circularly symmetric with common shape. Here, conditions (i) and (ii) ensure that the origin is the only (singular) point at which $\mathscr{E}$ vanishes to second order, while condition (iii) implies that the Euclidean norm of the gradient of any solution to (1) diverges to infinity as the circumference of $D(R)$ is approached. In this paper, we shall show that this assertion is invalid. Specifically, we shall reveal a class of functions f having the above properties, for which the corresponding eikonal equations have a bounded, non-circularly symmetric solution of class C^2. A companion paper will show how the above assumptions may be revised in order to rescue the assertion.

2. Solutions over Quadrants and Discs

The construction of non-circularly symmetric solutions to eikonal equations of the type described above will be divided into several steps. The graph of any such solution will take the form of a saddle having four regions of monotonicity spread out over four quadrants in the xy-plane determined by the lines $y = \pm x$. First, we shall construct a portion of a typical solution over the quadrant containing the positive x-halfaxis; the three remaining portions will easily be generated from this one. Next, we shall specify a class of functions f for which the portions over all four quadrants can be smoothly pasted together and shall describe the corresponding process of gluing. Finally, we shall discuss the differentiability properties of the solutions obtained.

We now undertake the first stage of the construction.

Theorem 1. *Let R be either a positive number or $+\infty$. Let f be a positive function of class C^2 on $(0, R)$ such that*

$$\lim_{r \to 0} f(r) = 0, \tag{3}$$

$$\lim_{r \to 0} \frac{f'(r)}{r} = 2, \tag{4}$$

and

$$r[f''(r) f(r) - (f'(r))^2] + f(r) f'(r) \geqslant 0 \tag{5}$$

for $0 < r < R$. Then there is a unique solution u of class C^2 to (1), *with $\mathscr{E}$ given by* (2), *defined over the quadrant*

$$Q_1(R) = \{(x, y) \in \mathbb{R}^2 : |y| < x,\ 0 < x < R,\ x^2 + y^2 < R^2\},$$

such that u is positive in the upper xy-halfplane and vanishes at the positive x-halfaxis. Moreover,

$$u(x, -y) = -u(x, y) \tag{6}$$

for each (x, y) in $Q_1(R)$.

Proof. Suppose that u is a solution of class C^2 to (1), with $\mathscr{E}$ as above, defined over $Q_1(R)$, that is positive in the upper xy-halfplane and vanishes at the positive x-halfaxis. Bearing in mind that $(r, \theta) \to (r \cos\theta, r \sin\theta)$ is a bijection between $(0, R) \times (-\pi/4, \pi/4)$ and $Q_1(R)$, set

$$v(r, \theta) = u(r \cos\theta, r \sin\theta)$$

for each $0<r<R$ and each $-\pi/4<\theta<\pi/4$. It is easily verified that v satisfies the equation

$$r^2\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{\partial v}{\partial \theta}\right)^2=r^2 f(r). \tag{7}$$

Since u vanishes at the positive x-halfaxis, we have $v(\rho, 0)=0$ for $0<\rho<R$. Moreover, $(\partial v/\partial r)(\rho, 0)=0$. This jointly with (7) yields $(\partial v/\partial\theta)(\rho, 0)= \pm\rho(f(\rho))^{1/2}$. Since u is positive in the upper xy-halfplane and vanishes at the positive x-halfaxis, it follows that $(\partial v/\partial\theta)(\rho, 0)\geqslant 0$. Hence, finally, $(\partial v/\partial\theta)(\rho, 0)=\rho(f(\rho))^{1/2}$.

For each $0<\rho<R$, let $t\to(r(t,\rho), \theta(t,\rho), w(t,\rho), p_r(t,\rho), p_\theta(t,\rho))$ be the solution of the characteristic system of equations associated with (7)

$$\begin{aligned}
&\text{(i)} && dr/dt=2r^2 p_r,\\
&\text{(ii)} && d\theta/dt=2p_\theta,\\
&\text{(iii)} && dw/dt=2(r^2p_r^2+p_\theta^2),\\
&\text{(iv)} && dp_r/dt=-2rp_r^2+2rf(r)+r^2f'(r),\\
&\text{(v)} && dp_\theta/dt=0,
\end{aligned} \tag{8}$$

that satisfies the initial conditions

$$\begin{aligned}
&\text{(i)} && r(0,\rho)=\rho,\\
&\text{(ii)} && \theta(0,\rho)=0,\\
&\text{(iii)} && w(0,\rho)=0,\\
&\text{(iv)} && p_r(0,\rho)=0,\\
&\text{(v)} && p_\theta(0,\rho)=\rho\sqrt{f(\rho)},
\end{aligned} \tag{9}$$

and is defined on a maximal interval. It is readily verified that $t\to(r(-t, \rho), -\theta(-t,\rho), -w(-t,\rho), -p_r(-t,\rho), p_\theta(-t,\rho))$ also satisfies the same system of equations and same initial conditions. Thus the maximal interval has a symmetric form $(-T_\rho, T_\rho)$ and, for each $-T_\rho<t<T_\rho$,

$$\begin{aligned}
&\text{(i)} && r(-t,\rho)=r(t,\rho),\\
&\text{(ii)} && \theta(-t,\rho)=-\theta(t,\rho),\\
&\text{(iii)} && w(-t,\rho)=-w(t,\rho),\\
&\text{(iv)} && p_r(-t,\rho)=-p_r(t,\rho),\\
&\text{(v)} && p_\theta(-t,\rho)=p_\theta(t,\rho).
\end{aligned} \tag{10}$$

Note that, for each $0<\rho<R$, the function $t \to (r(t, \rho), \theta(t, \rho), w(t, \rho), p_r(t, \rho), p_\theta(t, \rho))$ obeys

$$r^2 p_r^2 + p_\theta^2 = r^2 f(r); \tag{11}$$

this follows from (9) for $t=0$, and from [5, Lemma VI 8.1] for $t \neq 0$. Observe also that the initial conditions (9) are non-characteristic. In fact, for each $0<\rho<R$,

$$2\frac{\partial r}{\partial \rho}(0, \rho)\, p_\theta(0, \rho) - 2\frac{\partial \theta}{\partial \rho}(0, \rho)\, r^2(0, \rho)\, p_r(0, \rho) = 2\rho\sqrt{f(\rho)} \neq 0.$$

Thus

$$S = \left\{ (r(t, \rho), \theta(t, \rho), w(t, \rho), p_r(t, \rho), p_\theta(t, \rho)) : \right.$$

$$\left. (t, \rho) \in \bigcup_{0<\rho<R} (-T_\rho, T_\rho) \times \{\rho\} \right\}$$

is a (two-dimensional) surface of class C^1 (cf. [5, Theorem VI 9.1]). Let π be the projection defined by

$$\pi((r, \theta, w, p_r, p_\theta)) = (r, \theta).$$

As we shall see shortly, the restriction of π to S, $\pi|_S$, is one-to-one and its range coincides with $(0, R) \times (-\pi/4, \pi/4)$. Taking this temporarily for granted, we can now apply Cauchy's theory of characteristics to draw the following two conclusions: First, S coincides with

$$J(v) = \left\{ \left(r, \theta, v(r, \theta), \frac{\partial v}{\partial r}(r, \theta), \frac{\partial v}{\partial \theta}(r, \theta) \right) : (r, \theta) \in (0, R) \times \left(-\frac{\pi}{4}, \frac{\pi}{4} \right) \right\}$$

(cf. [5, Corollary VI 8.1]); this immediately yields the uniqueness of v and hence the uniqueness of u. Second, there exists a unique solution of class C^2 to (7) over $(0, R) \times (-\pi/4, \pi/4)$, still denoted by v, such that $S = J(v)$ (cf. [5, Theorem VI 9.1]). In the sequel, v will be used to generate the desired solution to (1).

Proceeding to establish the aforementioned properties of $\pi|_S$, notice that by (3), (4), and L'Hôpital's rule,

$$\lim_{r \to 0} \frac{f(r)}{r^2} = 1, \tag{12}$$

whence, still by (4),

$$\lim_{r \to 0} \frac{r f'(r)}{f(r)} = 2. \tag{13}$$

By (5), for $0<r<R$,

$$\left(\frac{rf'(r)}{f(r)}\right)'=\frac{r[f''(r)f(r)-(f'(r))^2]+f(r)f'(r)}{f^2(r)}\geqslant 0, \tag{14}$$

so the function $r\to rf'(r)/f(r)$ is non-decreasing. This jointly with (13) yields

$$\frac{rf'(r)}{f(r)}\geqslant 2 \tag{15}$$

for $0<r<R$. In particular, f' is positive and f is increasing.

In view of (11), (8iv) can equivalently be written as

$$\frac{dp_r}{dt}=\frac{2p_\theta^2}{r}+r^2f'(r).$$

Thus, since f' is positive, for each $0<\rho<R$ the function $t\to p_r(t,\rho)$ is increasing, and hence, by (9iv), $p_r(t,\rho)$ is positive for $0<t<T_\rho$.

By (8v) and (9v), for each $0<\rho<R$ and each $-T_\rho<t<T_\rho$,

$$p_\theta(t,\rho)=\rho\sqrt{f(\rho)} \tag{16}$$

and, by (8ii) and (9ii),

$$\theta(t,\rho)=2\rho\sqrt{f(\rho)}\,t. \tag{17}$$

Fix $0<\rho<R$. In view of (11), (16), and the positiveness of $p_r(t,\rho)$ for $0<t<T_\rho$, we have

$$p_r(t,\rho)=\frac{\sqrt{r^2(t,\rho)f(r(t,\rho))-\rho^2f(\rho)}}{r(t,\rho)} \tag{18}$$

for $0<t<T_\rho$. With this identity at hand, it is now easy to see that

$$t=\int_\rho^{r(t,\rho)}\frac{ds}{2s\sqrt{s^2f(s)-\rho^2f(\rho)}} \tag{19}$$

for $0\leqslant t<T_\rho$. Indeed, by (9i), the right-hand side of (19) vanishes for $t=0$ and, by (8i) and (18), its derivative with respect to t is equal to 1.

For any $s\geqslant 1$ and any $\rho>0$, set

$$g(s,\rho)=\frac{f(s\rho)}{f(\rho)}.$$

Using the identity

$$\frac{\partial g}{\partial \rho}(s,\rho)=\frac{g(s,\rho)}{\rho}\left[s\rho\,\frac{f'(s\rho)}{f(s\rho)}-\rho\,\frac{f'(\rho)}{f(\rho)}\right] \tag{20}$$

and the fact that the function $r \to rf'(r)/f(r)$ is non-decreasing, we infer that the function $\rho \to g(s, \rho)$ is non-decreasing. Moreover, by (12),

$$\lim_{\rho \to 0} g(s,\rho)=s^2. \tag{21}$$

Hence

$$g(s,\rho)\geqslant s^2. \tag{22}$$

In view of (17) and (19), if $0<\rho<R$ and $0\leqslant t<T_\rho$, then

$$\theta(t,\rho)=\int_1^{r(t,\rho)/\rho}\frac{ds}{s\sqrt{s^2 g(s,\rho)-1}}. \tag{23}$$

This together with (22) shows that

$$0\leqslant \theta(t,\rho)<\int_1^\infty \frac{ds}{s\sqrt{s^4-1}}=\frac{\pi}{4} \tag{24}$$

whenever $0<\rho<R$ and $0\leqslant t<T_\rho$.

We now prove that, for each $0<\rho<R$,

$$\lim_{t\to T_\rho} r(t,\rho)=R. \tag{25}$$

First note that, by virtue of (17) and (24), $T_\rho \leqslant \pi(8\rho)^{-1}(f(\rho))^{-1/2}$. In view of (8i) and (8iv), the finiteness of T_ρ implies that as $t \to T_\rho$ the curve $t \to (r(t,\rho),\ p_r(t,\rho))$ eventually leaves any compact subset of $(0, R)\times\mathbb{R}$ (cf. [5, Theorem II 3.1]). By (19), the function $t \to r(t,\rho)$ $(0\leqslant t<T_\rho)$ is increasing, so $\lim_{t\to T_\rho} r(t,\rho)$, whether it be finite or infinite, exists. If this were equal to a number r' smaller than R, then, taking into account (11) and the fact that f is increasing, we would have $p_r(t,\rho)\leqslant (f(r'))^{1/2}$ for $0<t<T_\rho$ and consequently the set $\{(r(t,\rho),\ p_r(t,\rho)) : 0<t<T_\rho\}$ would be contained in the compact set $[\rho, r']\times[0, (f(r'))^{1/2}]$, a contradiction.

Using (25) and the fact that for each $0<\rho<R$ the function $t\to r(t,\rho)$ $(0\leqslant t<T_\rho)$ is increasing, we see that if $0<\sigma<R$ and $0<\rho\leqslant\sigma$, then there exists a unique $t_{\sigma,\rho}\geqslant 0$ such that $r(t_{\sigma,\rho},\rho)=\sigma$. Clearly, by (19),

$$t_{\sigma,\rho}=\int_\rho^\sigma \frac{ds}{2s\sqrt{s^2 f(s)-\rho^2 f(\rho)}}=\frac{1}{2\rho\sqrt{f(\rho)}}\int_1^{\sigma/\rho}\frac{ds}{s\sqrt{s^2 g(s,\rho)-1}}$$

and hence, by (17),

$$\theta(t_{\sigma,\rho},\rho)=\int_1^{\sigma/\rho}\frac{ds}{s\sqrt{s^2 g(s,\rho)-1}}. \tag{26}$$

For each $0<\sigma<R$ the function $\rho\to\theta(t_{\sigma,\rho},\rho)$ $(0<\rho\leqslant\sigma)$ is decreasing. Indeed, since for each $s\geqslant 1$ the function $\rho\to g(s,\rho)$ is non-decreasing, it follows that if ρ_1 and ρ_2 satisfy $0<\rho_1<\rho_2\leqslant\sigma$, then

$$\begin{aligned}\theta(t_{\sigma,\rho_1},\rho_1)&=\int_1^{\sigma/\rho_1}\frac{ds}{s\sqrt{s^2 g(s,\rho_1)-1}}\\ &\geqslant\int_1^{\sigma/\rho_1}\frac{ds}{s\sqrt{s^2 g(s,\rho_2)-1}}\\ &>\int_1^{\sigma/\rho_2}\frac{ds}{s\sqrt{s^2 g(s,\rho_2)-1}}\\ &=\theta(t_{\sigma,\rho_2},\rho_2).\end{aligned}$$

In view of (22), the integrand in (26) is bounded above by the function $s^{-1}(s^4-1)^{-1/2}$ integrable over $(1,+\infty)$. Hence, first, for each $0<\sigma<R$, $\lim_{\rho\to\sigma}\theta(t_{\sigma,\rho},\rho)=0$, and, second, since the upper limit of integration in (26) diverges to infinity and, by (21), the integrand tends to $s^{-1}(s^4-1)^{-1/2}$ as $\rho\to 0$,

$$\lim_{\rho\to 0}\theta(t_{\sigma,\rho},\rho)=\int_1^{\infty}\frac{ds}{s\sqrt{s^4-1}}=\frac{\pi}{4}. \tag{27}$$

It is now clear that, for each $0<\sigma<R$,

$$\{\theta(t_{\sigma,\rho},\rho):0<\rho\leqslant\sigma\}=[0,\pi/4).$$

Noting that if $0<\sigma<R$ and $0<\rho\leqslant\sigma$, then

$$\pi(r(t_{\sigma,\rho},\rho),\theta(t_{\sigma,\rho},\rho),w(t_{\sigma,\rho},\rho),p_r(t_{\sigma,\rho},\rho),p_\theta(t_{\sigma,\rho},\rho))=(\sigma,\theta(t_{\sigma,\rho},\rho))$$

and, by (10),

$$\begin{aligned}&\pi(r(-t_{\sigma,\rho},\rho),\theta(-t_{\sigma,\rho},\rho),w(-t_{\sigma,\rho},\rho),p_r(-t_{\sigma,\rho},\rho),p_\theta(-t_{\sigma,\rho},\rho))\\ &\quad=(\sigma,-\theta(t_{\sigma,\rho},\rho)),\end{aligned}$$

we see that

$$\begin{aligned}\pi(S)&\supset\bigcup_{0<\rho<R}\{(\sigma,\theta(t_{\sigma,\rho})):0<\rho\leqslant\sigma\}\\ &\quad\cup\bigcup_{0<\rho<R}\{(\sigma,-\theta(t_{\sigma,\rho})):0<\rho\leqslant\sigma\}\\ &=(0,R)\times(-\pi/4,\pi/4).\end{aligned}$$

On the other hand, (10ii) and (24) imply the reverse inclusion. Hence $\pi(S)=(0, R)\times(-\pi/4, \pi/4)$.

To prove the injectiveness of $\pi|_S$, assume that

$$\begin{aligned} &\text{(i)} && r(t_1, \rho_1)=r(t_2, \rho_2), \\ &\text{(ii)} && \theta(t_1, \rho_1)=\theta(t_2, \rho_2) \end{aligned} \tag{28}$$

for some $0<\rho_1<R$, $0<\rho_2<R$, $-T_{\rho_1}<t_1<T_{\rho_1}$, and $-T_{\rho_2}<t_2<T_{\rho_2}$. By (17) and (28ii), either both t_1 and t_2 are non-negative or they both are negative. Since, in view of (10i) and (10ii), (28) remains valid if t_1 and t_2 are replaced by $-t_1$ and $-t_2$, respectively, we may assume without loss of generality that both t_1 and t_2 are non-negative. Suppose that $\rho_1\neq\rho_2$, say $\rho_1<\rho_2$. If we let $\sigma=r(t_1, \rho_1)=r(t_2, \rho_2)$, then $t_1=t_{\sigma, \rho_1}$ and $t_2=t_{\sigma, \rho_2}$ and now it is clear that (28ii) is incompatible with the fact that the function $\rho\to\theta(t_{\sigma, \rho}, \rho)$ $(0<\rho\leqslant\sigma)$ is decreasing. Thus $\rho_1=\rho_2$ and, further, by (17) and (28ii), $t_1=t_2$. The injectiveness of $\pi|_S$ follows.

Let v be the solution of class C^2 to (7) over $\pi(S)=(0, R)\times(-\pi/4, \pi/4)$ such that $S=J(v)$. Observe that $v(r, 0)=0$ for each $0<r<R$ and $-v(r, -\theta)=v(r, \theta)>0$ for each $0<r<R$ and each $0<\theta<\pi/4$. In fact, the first relation follows from (9iii) upon noting that $v(r, 0)=w(0, r)$ for each $0<r<R$. To prove the second, given $0<r<R$ and $0<\theta<\pi/4$, let $0<\rho<R$ be such that $\theta=\theta(t_{r, \rho}, \rho)$. Of course, $t_{\sigma, \rho}$ is positive. By (8iii), (9iii), and (11),

$$v(r, \theta)=w(t_{r, \rho}, \rho)=2\int_0^{t_{r, \rho}} r^2(s, \rho)\, f(r(s, \rho))\, ds>0;$$

by (10ii), $-\theta=\theta(-t_{r, \rho}, \rho)$; and finally, by (10iii),

$$v(r, -\theta)=w(-t_{r, \rho}, \rho)=-w(t_{r, \rho}, \rho)=-v(r, \theta).$$

Now it is easy to verify that setting

$$u(x, y)=v\left(\sqrt{x^2+y^2}, \arctan\frac{y}{x}\right)\qquad((x, y)\in Q_1(R))$$

defines the required solution to (1). The proof is complete.

Proceeding to the next stage of the construction, let R be either a positive number or $+\infty$, and let

$$Q_2(R)=\{(x, y)\in\mathbb{R}^2 : |x|<y,\ 0<y<R,\ x^2+y^2<R^2\},$$

$$Q_3(R)=\{(x, y)\in\mathbb{R}^2 : |y|<-x, -R<x<0,\ x^2+y^2<R^2\},$$

$$Q_4(R)=\{(x, y)\in\mathbb{R}^2 : |x|<-y, -R<y<0,\ x^2+y^2<R^2\}.$$

Given a positive function f of class C^2 on $(0, R)$ satisfying (3), (4), and (5), let u be the solution to (1), with $\mathscr{E}$ given by (2), defined over $Q_1(R)$ that has the properties stated in Theorem 1. Let

$$U(x, y) = \begin{cases} u(x, y), & \text{if } (x, y) \in Q_1(R); \\ u(y, x), & \text{if } (x, y) \in Q_2(R); \\ u(-x, -y), & \text{if } (x, y) \in Q_3(R); \\ u(-y, -x), & \text{if } (x, y) \in Q_4(R); \\ \int_0^{\sqrt{2}\,|x|} \sqrt{f(\tau)}\, d\tau, & \text{if } -R < x = y < R; \\ -\int_0^{\sqrt{2}\,|x|} \sqrt{f(\tau)}\, d\tau, & \text{if } -R < x = -y < R. \end{cases}$$

We have the following.

THEOREM 2. *Let R be either a positive number or $+\infty$. Let f be a positive function of class C^2 over $(0, R)$ satisfying* (3), (4), *and* (5). *Suppose, moreover, that for some $0 < r_0 < R$, f is of class C^4 over $[0, r_0)$ and of class C^5 over $(0, r_0)$, and that $f^{(5)}$ is bounded in $(0, r_0)$. Then U is a solution to* (1), *with $\mathscr{E}$ given by* (2), *of class C^1 over $D(R)$ and of class C^2 over $D(R) \setminus \{(0, 0)\}$.*

Proof. First note that if we show that the function U is a solution of class C^2 to (1) over $D(R) \setminus \{(0, 0)\}$, then U being a solution of class C^1 to (1) over $D(R)$ follows immediately from the fact that U vanishes at both the x-axis and y-axis, and $\lim_{r \to 0} f(r) = 0$. In proving that U is a solution of class C^2 to (1) over $D(R) \setminus \{(0, 0)\}$, we confine ourselves to consideration of U over the semidisc

$$S_{12}(R) = Q_1(R) \cup Q_2(R) \cup \{(x, x) \in \mathbb{R}^2 : 0 < x < R\}.$$

The proof of the assertion that U is a solution of class C^2 to (1) over the three semidiscs obtained by rotating $S_{12}(R)$ around the origin by angles $\pi/2$, π, and $3\pi/2$, respectively, is analogous and will be omitted.

Bearing in mind that $(r, \theta) \to (r \cos\theta, r \sin\theta)$ is a bijection between $(0, R) \times (-\pi/4, 3\pi/4)$ and $S_{12}(R)$, set

$$V(r, \theta) = U(r \cos\theta, r \sin\theta)$$

for each $0 < r < R$ and each $-\pi/4 < \theta < 3\pi/4$. To prove that U is a solution of class C^2 to (1) over $S_{12}(R)$, it suffices to show that V is a solution of class C^2 to (7) over $(0, R) \times (-\pi/4, 3\pi/4)$. Retaining the notation from the proof to Theorem 1, denote by v the solution of class C^2 to (7) such that $S=J(v)$. One verifies at once that, for each $0<r<R$,

$$V(r,\theta)=\begin{cases} v(r,\theta), & \text{if} \quad -\pi/4<\theta<\pi/4; \\ v\left(r,\dfrac{\pi}{2}-\theta\right), & \text{if} \quad \pi/4<\theta<3\pi/4; \\ \displaystyle\int_0^r \sqrt{f(\tau)}\,d\tau, & \text{if} \quad \theta=\pi/4. \end{cases} \tag{29}$$

In particular, V is a solution of class C^2 to (7) over $(0,R)\times[(-\pi/4, 3\pi/4)\setminus\{\pi/4\}]$. We now prove that V being a solution of class C^2 to (7) $(0,R)\times(-\pi/4, 3\pi/4)$ is a consequence of the assertion that, for each $0<r<R$, the function $\theta\to V(r,\theta)$ is of class C^2 over $(-\pi/4, 3\pi/4)$.

Assume for now the truth of the above assertion. Fix arbitrarily $0<\sigma<R$. We first show that the function $\eta\to(\sigma^2 f(\sigma)-((\partial V/\partial\theta)(\sigma,\eta))^2)^{1/2}$ is positive and of class C^1 over $(0,\pi/2)$.

If $0<\eta<\pi/4$, then, with the notation from the proof of Theorem 1, there exists $0<\rho<R$ such that $(\partial v/\partial r)(\sigma,\eta)=p_r(t_{\sigma,\rho},\rho)$. Of course, $t_{\sigma,\rho}$ is positive and hence $p_r(t_{\sigma,\rho},\rho)$ is also positive. Therefore, by (7),

$$\sqrt{\sigma^2 f(\sigma)-((\partial V/\partial\theta)(\sigma,\eta))^2}=\sigma\,\frac{\partial v}{\partial r}\,(\sigma,\eta)>0.$$

If $\pi/4<\eta<\pi/2$, then

$$\sqrt{\sigma^2 f(\sigma)-((\partial V/\partial\theta)(\sigma,\eta))^2}=\sigma\,\frac{\partial v}{\partial r}\left(\sigma,\frac{\pi}{2}-\eta\right)>0,$$

and, as by (29) we have that $(\partial V/\partial\theta)(\sigma,\pi/4)=0$,

$$\sqrt{\sigma^2 f(\sigma)-((\partial V/\partial\theta)(\sigma,\pi/4))^2}=\sigma\sqrt{f(\sigma)}>0.$$

Now that the function $\eta\to(\sigma^2 f(\sigma)-((\partial V/\partial\theta)(\sigma,\eta))^2)^{1/2}$ is positive and the function $\eta\to V(\sigma,\eta)$ is of class C^2, we see that the first of these functions is of class C^1.

For each $0<\eta<\pi/2$, let $t\to(\bar r(t,\eta), \bar\theta(t,\eta), \bar w(t,\eta), \bar p_r(t,\eta), \bar p_\theta(t,\eta))$ be the solution to (8) that satisfies the initial conditions

$$\begin{aligned} &\bar r(0,\eta)=\sigma, \\ &\bar\theta(0,\eta)=\eta, \\ &\bar w(0,\eta)=V(\sigma,\eta), \\ &\bar p_r(0,\eta)=\sigma^{-1}\sqrt{\sigma^2 f(\sigma)-((\partial V/\partial\theta)(\sigma,\eta))^2}, \\ &\bar p_\theta(0,\eta)=(\partial V/\partial\theta)(\sigma,\eta), \end{aligned} \tag{30}$$

and is defined over a maximal interval I_η. Since, for each $0<\eta<\pi/2$,

$$2\,\frac{\partial \bar{r}}{\partial \eta}(0,\eta)\,\bar{p}_\theta(0,\eta)-2\,\frac{\partial \bar{\theta}}{\partial \eta}(0,\eta)\,\bar{r}^2(0,\eta)\,\bar{p}_r(0,\eta)$$
$$=-2\sigma\sqrt{\sigma^2 f(\sigma)-((\partial V/\partial\theta)(\sigma,\eta))^2}<0,$$

the initial conditions (30) are non-characteristic. As these initial conditions are also of class C^1, we see that

$$\Sigma=\left\{(\bar{r}(t,\eta),\bar{\theta}(t,\eta),\bar{w}(t,\eta),\bar{p}_r(t,\eta),\bar{p}_\theta(t,\eta)):(t,\eta)\in\bigcup_{0<\eta<\pi/2} I_\eta\times\{\eta\}\right\}$$

is a surface of class C^1. Let

$$D_1=\left[(0,\sigma)\times\left(0,\frac{\pi}{4}\right)\right]\cup\left\{(\xi,\eta):\sigma\leqslant\xi<R,\ \theta(t_{\xi,\sigma},\sigma)<\eta<\frac{\pi}{4}\right\},$$

$$D_2=\left[(0,\sigma)\times\left(\frac{\pi}{4},\frac{\pi}{2}\right)\right]\cup\left\{(\xi,\eta):\sigma\leqslant\xi<R,\ \frac{\pi}{4}<\eta<\frac{\pi}{2}-\theta(t_{\xi,\sigma},\sigma)\right\},$$

and

$$D=D_1\cup D_2\cup\left[(0,R)\times\left\{\frac{\pi}{4}\right\}\right].$$

Figure 1 displays the images Δ_1 and Δ_2 of D_1 and D_2 under the mapping $(r,\theta)\to(r\cos\theta,\,r\sin\theta)$. A moment's reflection reveals that

$$\left\{(\bar{r}(t,\eta),\bar{\theta}(t,\eta),\bar{w}(t,\eta),\bar{p}_r(t,\eta),\bar{p}_\theta(t,\eta)):(t,\eta)\in\bigcup_{\eta\in(0,\pi/2)\setminus\{\pi/4\}} I_\eta\times\{\}\right\}$$
$$=\left\{\left(r,\theta,V(r,\theta),\frac{\partial V}{\partial r}(r,\theta),\frac{\partial V}{\partial\theta}(r,\theta)\right):(r,\theta)\in D_1\cup D_2\right\}$$

and that

$$\left\{\left(\bar{r}\left(t,\frac{\pi}{4}\right),\bar{\theta}\left(t,\frac{\pi}{4}\right),\bar{w}\left(t,\frac{\pi}{4}\right),\bar{p}_r\left(t,\frac{\pi}{4}\right),\bar{p}_\theta\left(t,\frac{\pi}{4}\right)\right):t\in I_{\pi/4}\right\}$$
$$=\left\{\left(r,\frac{\pi}{4},\int_0^r\sqrt{f(\tau)}\,d\tau,\sqrt{f(r)},0\right):0<r<R\right\}.$$

Accordingly, $\pi|_\Sigma$ is one-to-one and $\pi(\Sigma)=D$, and so there exists a solution W of class C^2 to (7) over D such that

$$\Sigma=\left\{\left(r,\theta,W(r,\theta),\frac{\partial W}{\partial r}(r,\theta),\frac{\partial W}{\partial\theta}(r,\theta)\right):(r,\theta)\in D\right\}.$$

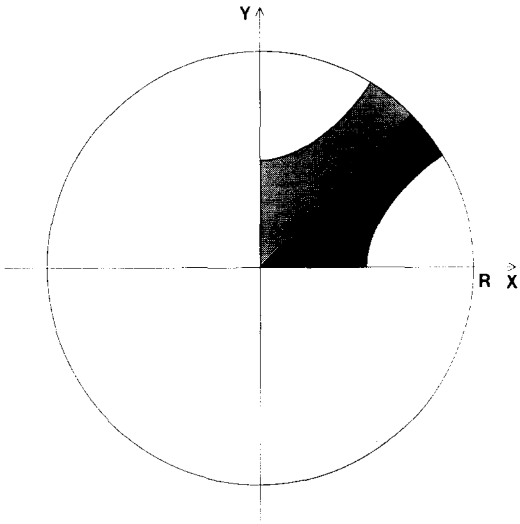

FIGURE 1

Of course, W coincides with V on $D_1 \cup D_2$. Since, for each $0<r<R$, the function $\theta \to V(r, \theta)$ $(0<\theta<\pi/2)$ is continuous, W actually coincides with V on D. Now it is clear that V is a solution of class C^2 to (7) over $(0, R) \times (-\pi/4, 3\pi/4)$.

In view of (29), the assertion whose validity we assumed will be established once we show that, for each $0<r<R$,

$$\lim_{\theta \to \pi/4} v(r, \theta) = \int_0^r \sqrt{f(\tau)}\, d\tau, \tag{31}$$

$$\lim_{\theta \to \pi/4} \frac{\partial v}{\partial \theta}(r, \theta) = 0, \tag{32}$$

and that $\lim_{\theta \to \pi/4} (\partial^2 v/\partial \theta^2)(r, \theta)$ exists. To prove the existence of the latter limit, we shall demonstrate that

$$\lim_{\theta \to \pi/4} \frac{\partial^2 v}{\partial \theta^2}(r, \theta) = -2\left[\frac{1}{r\sqrt{f(r)}} + \int_0^r \frac{\tau f'(\tau) - 2f(\tau)}{2\tau^2 (f(\tau))^{3/2}}\, d\tau\right]^{-1}. \tag{33}$$

Given $0<r<R$ and $0<\theta<\pi/4$, let $0<\rho<r$ be such that $\theta = \theta(t_{r,\rho}, \rho)$. By (26), if we let

$$h(r, \rho) = \int_1^{r/\rho} \frac{ds}{s\sqrt{s^2 g(s, \rho) - 1}},$$

then, clearly, $\theta = h(r, \rho)$. Of course, $v(r, \theta) = w(t_{r,\rho}, \rho)$. Moreover, by (16),

$$\frac{\partial v}{\partial \theta}(r, \theta) = p_\theta(t_{r,\rho}, \rho) = \rho\sqrt{f(\rho)},$$

and, hence,

$$\frac{\partial^2 v}{\partial \theta^2}(r, \theta) = \frac{2f(\rho) + \rho f'(\rho)}{2\sqrt{f(\rho)}(\partial h/\partial \rho)(r, \rho)}.$$

Now, in view of (27), (31) reduces to the identity

$$\lim_{\rho \to 0} w(t_{r,\rho}, \rho) = \int_0^r \sqrt{f(\tau)}\, d\tau, \tag{34}$$

(32) amounts to the obvious equality $\lim_{\rho \to 0} \rho(f(\rho))^{1/2} = 0$, and, in view of (4) and (12), (33) reduces to the identity

$$\lim_{\rho \to 0} \frac{1}{\rho}\frac{\partial h}{\partial \rho}(r, \rho) = -\frac{1}{r\sqrt{f(r)}} - \int_0^r \frac{\tau f'(\tau) - 2f(\tau)}{2\tau^2(f(\tau))^{3/2}}\, d\tau. \tag{35}$$

To establish (34), note that by (8i), (8iii), (9i), (9iii), (11), and (18), for each $0 < \rho < R$ and each $0 < t < T_\rho$,

$$\begin{aligned} w(t, \rho) &= 2\int_0^t r^2(s, \rho) f(r(s, \rho))\, ds = \int_0^t \frac{f(r(s, \rho))(\partial r/\partial s)(s, \rho)}{p_r(s, \rho)}\, ds \\ &= \int_0^t \frac{r(s, \rho) f(r(s, \rho))(\partial r/\partial s)(s, \rho)}{\sqrt{r^2(s, \rho) f(r(s, \rho)) - \rho^2 f(\rho)}}\, ds = \int_\rho^{r(t, \rho)} \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau) - \rho^2 f(\rho)}}\, d\tau. \end{aligned}$$

Thus (34) will follow once we show that, for each $0 < r < R$,

$$\lim_{\rho \to 0} \int_\rho^r \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau) - \rho^2 f(\rho)}}\, d\tau = \int_0^r \sqrt{f(\tau)}\, d\tau. \tag{36}$$

We first prove that

$$\lim_{\delta \to 0} \sup\left\{\int_\rho^\delta \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau) - \rho^2 f(\rho)}}\, d\tau : 0 < \rho < \delta\right\} = 0. \tag{37}$$

By (12), there exist $c_1 > 0$, $c_2 > 0$, and $0 < \delta_0 < R$ such that if $0 < \tau \leqslant \delta_0$, then

$$c_1 \tau^2 \leqslant f(\tau) \leqslant c_2 \tau^2. \tag{38}$$

Let ρ and δ be such that $0<\rho<\delta<\delta_0$. Then, by (22) and (38),

$$\begin{aligned}\int_\rho^\delta \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau &\leqslant \int_\rho^\delta \frac{c_2\tau^3}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau \\ &= \frac{c_2\rho^3}{\sqrt{f(\rho)}}\int_1^{\delta/\rho}\frac{s^3}{\sqrt{s^2 g(s,\rho)-1}}\,ds \\ &\leqslant \frac{c_2\rho^3}{\sqrt{f(\rho)}}\int_1^{\delta/\rho}\frac{s^3}{\sqrt{s^4-1}}\,ds \\ &= \frac{c_2\rho^3}{2\sqrt{f(\rho)}}\sqrt{(\delta/\rho)^4-1}\leqslant\frac{c_2\rho\delta^2}{2\sqrt{f(\rho)}}\leqslant\frac{c_2\delta^2}{\sqrt{c_1}},\end{aligned}$$

proving (37).

As the function $\tau\to\tau^2 f(\tau)$ is increasing and tends to zero as $\tau\to 0$, it is clear that for each $0<\delta<R$ there exists $\rho_0=\rho_0(\delta)$ such that if $0<\rho<\rho_0$ and $\delta\leqslant\tau<R$, then $4\rho^2 f(\rho)/3\leqslant\tau^2 f(\tau)$. Note that the latter inequality implies that $\tau^2 f(\tau)/4\leqslant\tau^2 f(\tau)-\rho^2 f(\rho)$ and next that $\tau f(\tau)(\tau^2 f(\tau)-\rho^2 f(\rho))^{-1/2}\leqslant 2(f(\tau))^{1/2}$. Accordingly, we can apply Lebesgue's dominated convergence theorem to conclude that if $0<\delta<r$, then

$$\lim_{\rho\to 0}\int_\delta^r \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau=\int_\delta^r\sqrt{f(\tau)}\,d\tau. \tag{39}$$

Let there be given $0<r<R$ and $\varepsilon>0$. Using (37), choose $0<\delta<r$ so that

$$\sup\left\{\int_\rho^\delta \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau : 0<\rho<\delta\right\}<\frac{\varepsilon}{3}$$

and

$$\int_0^\delta\sqrt{f(\tau)}\,d\tau<\frac{\varepsilon}{3}.$$

Next, using (39), select $0<\rho_1<\delta$ so that, for each $0<\rho<\rho_1$,

$$\left|\int_\delta^r \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau-\int_\delta^r\sqrt{f(\tau)}\,d\tau\right|<\frac{\varepsilon}{3}.$$

Then, for each $0<\rho<\rho_1$,

$$\begin{aligned}&\left|\int_\rho^r \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau-\int_0^r\sqrt{f(\tau)}\,d\tau\right| \\ &\quad\leqslant\int_\rho^\delta \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau+\left|\int_\delta^r \frac{\tau f(\tau)}{\sqrt{\tau^2 f(\tau)-\rho^2 f(\rho)}}\,d\tau-\int_\delta^r\sqrt{f(\tau)}\,d\tau\right| \\ &\qquad+\int_0^\delta\sqrt{f(\tau)}\,d\tau<\varepsilon,\end{aligned}$$

proving (36).

To establish (35), fix $0<r<R$ and note that, for each $0<\rho<r$,

$$\frac{1}{\rho}\frac{\partial h}{\partial \rho}(r,\rho)=-\frac{1}{2\rho}\int_1^{r/\rho}\frac{s(\partial g/\partial\rho)(s,\rho)}{(s^2g(s,\rho)-1)^{3/2}}\,ds-\frac{\sqrt{f(\rho)}}{\rho\sqrt{r^2f(r)-\rho^2f(\rho)}}.$$

In view of (12), all that we need to show is that

$$\lim_{\rho\to 0}\frac{1}{\rho}\int_1^{r/\rho}\frac{s(\partial g/\partial\rho)(s,\rho)}{(s^2g(s,\rho)-1)^{3/2}}\,ds=\int_0^r\frac{\tau f'(\tau)-2f(\tau)}{\tau^2(f(\tau))^{3/2}}\,d\tau. \tag{40}$$

Observe that if $0<\rho<r$, then

$$\frac{1}{\rho}\int_1^{r/\rho}\frac{s(\partial g/\partial\rho)(s,\rho)}{(s^2g(s,\rho)-1)^{3/2}}\,ds=\frac{\rho}{\sqrt{f(\rho)}}\int_\rho^r\frac{\tau(\tau f'(\tau)\,f(\rho)\,\rho^{-2}-f(\tau)\,f'(\rho)\,\rho^{-1})}{(\tau^2f(\tau)-\rho^2f(\rho))^{3/2}}\,d\tau. \tag{41}$$

Since the function $\tau\to\tau f'(\tau)/f(\tau)$ is non-decreasing, the integrand in the right-hand side is non-negative. Hence, by (4), (12), and Fatou's lemma,

$$\int_0^r\frac{\tau f'(\tau)-2f(\tau)}{\tau^2(f(\tau))^{3/2}}\,d\tau\leqslant\liminf_{\rho\to 0}\frac{1}{\rho}\int_1^{r/\rho}\frac{s(\partial g/\partial\rho)(s,\rho)}{(s^2g(s,\rho)-1)^{3/2}}\,ds,$$

and so if

$$\int_0^r\frac{\tau f'(\tau)-2f(\tau)}{\tau^2(f(\tau))^{3/2}}\,d\tau=+\infty,$$

then (40) clearly holds.

Suppose now that

$$\int_0^r\frac{\tau f'(\tau)-2f(\tau)}{\tau^2(f(\tau))^{3/2}}\,d\tau<+\infty. \tag{42}$$

If we let $h(\tau)=\tau f'(\tau)/f(\tau)$ for each $0<\tau<R$, then, by (12) and (42),

$$\int_0^r\frac{h(\tau)-2}{\tau^3}\,d\tau<+\infty.$$

Integrating by parts, we have that, for each $0<\delta<r$,

$$2\int_\delta^r\frac{h(\tau)-2}{\tau^3}\,d\tau+\frac{h(r)-2}{r^2}=\frac{h(\delta)-2}{\delta^2}+\int_\delta^r\frac{h'(\tau)}{\tau^2}\,d\tau.$$

Hence, by (15),

$$2\int_\delta^r\frac{h(\tau)-2}{\tau^3}\,d\tau+\frac{h(r)-2}{r^2}\geqslant\int_\delta^r\frac{h'(r)}{\tau^2}\,d\tau$$

and next, by (14) and Lebesgue's monotone convergence theorem,

$$2\int_0^r \frac{h(\tau)-2}{\tau^3}\,d\tau+\frac{h(r)-2}{r^2}\geqslant\int_0^r \frac{h'(\tau)}{\tau^2}\,d\tau.$$

Thus

$$\int_0^r \frac{h'(\tau)}{\tau^2}\,d\tau<+\infty,$$

showing that

$$\lim_{\delta\to 0}\int_0^\delta \frac{h'(\tau)}{\tau^2}\,d\tau=0. \tag{43}$$

By (12) and (42),

$$\int_0^r \frac{\tau f'(\tau)-2f(\tau)}{\tau^5}\,d\tau<+\infty, \tag{44}$$

and by L'Hôpital's rule,

$$\lim_{\tau\to 0}\frac{\tau f'(\tau)-2f(\tau)}{\tau^3}=\frac{f'''(0)}{6}$$

Hence

$$f'''(0)=0. \tag{45}$$

Applying L'Hôpital's rule once again, we obtain

$$\lim_{\tau\to 0}\frac{\tau f'(\tau)-2f(\tau)}{\tau^4}=\frac{f^{(4)}(0)}{12}$$

which, together with (44), yields

$$f^{(4)}(0)=0. \tag{46}$$

Using (4), (12), (45), (46), the fact that $f^{(5)}$ is bounded in $(0, r_0)$, and Taylor's formula, we now see that there exist bounded functions g_1, g_2, and g_3 on $(0, r_0)$ such that, for each $0<\tau<r_0$,

$$f(\tau)=\tau^2+g_1(\tau)\,\tau^5,$$
$$f'(\tau)=2\tau+g_2(\tau)\,\tau^4,$$
$$f''(\tau)=2+g_3(\tau)\,\tau^3.$$

Hence, if $0<\tau<r_0$, then

$$\begin{aligned}&\tau[f''(\tau)\,f(\tau)-(f'(\tau))^2]+f(\tau)\,f'(\tau)\\&\quad=[4g_1(\tau)+g_3(\tau)-3g_2(\tau)+(g_1(\tau)\,g_2(\tau)+g_1(\tau)\,g_3(\tau)-g_2^2(\tau))\,\tau^3]\,\tau^6.\end{aligned}$$

This together with (12) shows that

$$\lim_{\tau\to 0}\frac{h'(\tau)}{\tau}=0. \tag{47}$$

We now prove that

$$\lim_{\delta\to 0}\sup\left\{\frac{1}{\rho}\int_1^{\delta/\rho}\frac{s(\partial g/\partial\rho)(s,\rho)}{(s^2g(s,\rho)-1)^{3/2}}\,ds : 0<\rho<\delta\right\}=0. \tag{48}$$

With δ_0 such that (38) holds for $0<\tau<\delta_0$, let ρ and δ be such that $0<\rho<\delta<\delta_0$. Then, by (20) and (22),

$$\begin{aligned}\frac{1}{\rho}\int_1^{\delta/\rho}\frac{s(\partial g/\partial\rho)(s,\rho)}{(s^2g(s,\rho)-1)^{3/2}}\,ds&=\frac{1}{\rho^2f(\rho)}\int_1^{\delta/\rho}\frac{sf(s\rho)(h(s\rho)-h(\rho))}{(s^2g(s,\rho)-1)^{3/2}}\,ds\\&\leqslant\frac{c_2}{c_1\rho^2}\int_1^{\delta/\rho}\frac{s^3(h(s\rho)-h(\rho))}{(s^4-1)^{3/2}}\,ds.\end{aligned} \tag{49}$$

Integrating by parts and using the fact that the function $\tau\to h(\tau)$ is non-decreasing, we obtain

$$\begin{aligned}\int_1^{\delta/\rho}\frac{s^3(h(s\rho)-h(\rho))}{(s^4-1)^{3/2}}\,ds&=\frac{h(\rho)-h(\delta)}{2\sqrt{(\delta/\rho)^4-1}}+\int_1^{\delta/\rho}\frac{\rho h'(s\rho)}{2\sqrt{s^4-1}}\,ds\\&\leqslant\int_\rho^\delta\frac{h'(\tau)}{2\sqrt{(\tau/\rho)^4-1}}\,d\tau.\end{aligned} \tag{50}$$

In view of (14),

$$\begin{aligned}\int_\rho^\delta\frac{h'(\tau)}{\sqrt{(\tau/\rho)^4-1}}\,d\tau&=\int_\rho^{\min(2\rho,\delta)}\frac{h'(\tau)}{\sqrt{(\tau/\rho)^4-1}}\,d\tau\\&\quad+\int_{\min(2\rho,\delta)}^\delta\frac{h'(\tau)}{\sqrt{(\tau/\rho)^4-1}}\,d\tau.\end{aligned} \tag{51}$$

Of course,

$$\begin{aligned}\int_\rho^{\min(2\rho,\delta)}\frac{h'(\tau)}{\sqrt{(\tau/\rho)^4-1}}\,d\tau&\leqslant\max\left\{\frac{h'(\tau)}{\tau}:\rho\leqslant\tau\leqslant\delta\right\}\int_\rho^{2\rho}\frac{\tau}{\sqrt{(\tau/\rho)^4-1}}\,d\tau\\&\leqslant c_3\rho^2\sup\left\{\frac{h'(\tau)}{\tau}:0<\tau\leqslant\delta\right\},\end{aligned} \tag{52}$$

where $c_3 = \int_1^2 u(u^4-1)^{-1/2}\,du$. Since $(x^4-1)^{-1/2} < 2x^{-2}$ for $x > 2$, we see that if $2\rho < \delta$, then

$$\int_{\min(2\rho,\,\delta)}^{\delta} \frac{h'(\tau)}{\sqrt{(\tau/\rho)^4-1}}\,d\tau \leqslant 2\rho^2 \int_{2\rho}^{\delta} \frac{h'(\tau)}{\tau^2}\,d\tau \leqslant 2\rho^2 \int_0^{\delta} \frac{h'(\tau)}{\tau^2}\,d\tau;$$

if $2\rho \geqslant \delta$, then clearly

$$\int_{\min(2\rho,\,\delta)}^{\delta} \frac{h'(\tau)}{\sqrt{(\tau/\rho)^4-1}}\,d\tau = 0$$

and so in either case

$$\int_{\min(2\rho,\,\delta)}^{\delta} \frac{h'(\tau)}{\sqrt{(\tau/\rho)^4-1}}\,d\tau \leqslant 2\rho^2 \int_0^{\delta} \frac{h'(\tau)}{\tau^2}\,d\tau. \tag{53}$$

Comparing (49)–(53), we finally obtain

$$\begin{aligned}&\frac{1}{\rho}\int_1^{\delta/\rho} \frac{s(\partial g/\partial\rho)(s,\rho)}{(s^2 g(s,\rho)-1)^{3/2}}\,ds\\ &\quad\leqslant \frac{c_2}{c_1}\left(\frac{c_3}{2}\sup\left\{\frac{h'(\tau)}{\tau} : 0<\tau\leqslant\delta\right\} + \int_0^{\delta} \frac{h'(\tau)}{\tau^2}\,d\tau\right).\end{aligned}$$

Now (48) follows upon applying (43) and (47)

Using (4) and (12), and applying Lebesgue's dominated convergence theorem along the same lines as in the proof of (39), we find that, for $0 < \delta < r$,

$$\begin{aligned}&\lim_{\rho\to 0} \frac{\rho}{\sqrt{f(\rho)}} \int_{\delta}^{r} \frac{\tau(\tau f'(\tau)\,f(\rho)\,\rho^{-2} - f(\tau)\,f'(\rho)\,\rho^{-1})}{(\tau^2 f(\tau) - \rho^2 f(\rho))^{3/2}}\,d\tau\\ &\quad= \int_{\delta}^{r} \frac{\tau f'(\tau) - 2f(\tau)}{\tau^2 (f(\tau))^{3/2}}\,d\tau. \end{aligned}\tag{54}$$

Now, using the same final argument as in the proof of (36), we see that (41), (42), (48), and (54) imply (40). The proof is complete.

Having proved Theorem 2, it is natural to inquire as to whether or not the solution U is of class C^2 over the entire disc $D(R)$. The following theorem specifies certain conditions on the function f that must be met for the answer to be in the affirmative.

THEOREM 3. *Let R be either a positive number or $+\infty$. Let f be a positive function of class C^2 over $(0, R)$ and, for some $0 < r_0 < R$, of class C^4*

over $[0, r_0)$ *satisfying* (3), (4), *and* (5). *Suppose that* U *is of class* C^2 *over* $D(R)$. *Then* $f'''(0) = f^{(4)}(0) = 0$.

Proof. Note first that $U(x, 0) = U(0, x) = 0$ for $-R < x < R$. Hence

$$\frac{\partial U}{\partial x}(0,0) = \frac{\partial U}{\partial y}(0,0) = \frac{\partial^2 U}{\partial x^2}(0,0) = \frac{\partial^2 U}{\partial y^2}(0,0) \tag{55}$$

and, further, by (12),

$$\frac{\partial^2 U}{\partial x\,\partial y}(0,0) = \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{U(x,x)}{x^2} = \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{1}{x^2} \int_0^{\sqrt{2}\,|x|} \sqrt{f(\tau)}\,d\tau = 1. \tag{56}$$

Retaining the notation from the proof of Theorem 2, one easily verifies that, for each $0 < r < R$ and each $-\pi/4 < \theta < 3\pi/4$,

$$\begin{aligned} \frac{\partial^2 V}{\partial \theta^2}(r,\theta) + r\frac{\partial V}{\partial r}(r,\theta) = r^2 \sin^2\theta\, & \frac{\partial^2 U}{\partial x^2}(r\cos\theta, r\sin\theta) \\ & - r^2 \sin 2\theta\, \frac{\partial^2 U}{\partial x\,\partial y}(r\cos\theta, r\sin\theta) \\ & + r^2\cos^2\theta\, \frac{\partial^2 U}{\partial y^2}(r\cos\theta, r\sin\theta). \end{aligned}$$

Hence, by (12), (55), (56), and the identity $(\partial V/\partial r)(r, \pi/4) = (f(r))^{1/2}$ that follows immediately from (29), we obtain

$$\lim_{r \to 0} \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}\left(r, \frac{\pi}{4}\right) = -1 - \frac{\partial^2 U}{\partial x\,\partial y}(0,0) = -2. \tag{57}$$

Proceeding by *reductio ad absurdum*, suppose that either $f'''(0)$ or $f^{(4)}(0)$ is non-zero. Then, as inspection of the proof to Theorem 2 reveals, for each $0 < r < R$,

$$\int_0^r \frac{\tau f'(\tau) - 2f(\tau)}{\tau^2 (f(\tau))^{3/2}}\,d\tau = +\infty$$

and, hence, by (33), $(\partial^2 V/\partial\theta^2)(r, \pi/4) = 0$. This obviously contradicts (57). The proof is complete.

We conclude this section with a simple sufficient condition for U to be of class C^2 over $D(R)$.

THEOREM 4. *Let* R *be either a positive number or* $+\infty$. *Let* f *be a positive function that is of class* C^2 *over* $(0, R)$, *satisfies* (5), *and, for some* $0 < r_0 < R$, $f(r) = r^2$ *whenever* $0 \leqslant r < r_0$. *Then* U *is of class* C^2 *over* $D(R)$.

Proof. By the uniqueness part of Theorem 1, we have that $U(x, y) = xy$ for each (x, y) in $D(r_0)$. Hence U is of class C^2 over $D(r_0)$. Now the theorem follows upon applying Theorem 2.

3. Refinements

In this section, we specify certain classes of functions f to which the results of the previous section are applicable. One of these classes will be used to generate a counterexample to Bruss' assertion mentioned in the Introduction.

THEOREM 5. *Let R be a positive number. Let $g\colon (0, R) \to [0, 1)$ be a function of class C^2 such that g' and g'' are non-negative, g' is bounded in $(0, r_0)$ for some $0 < r_0 < R$, and $\lim_{r\to 0} g(r) = 0$. Then the function f defined by*

$$f(r) = \frac{r^2}{1 - g(r)} \qquad (0 < r < R) \tag{58}$$

is of class C^2 and satisfies (3), (4), and (5).

Proof. Obviously, f is of class C^2 and satisfies (3). Since, for each $0 < r < R$,

$$f'(r) = \frac{2r}{1 - g(r)} + \frac{r^2 g'(r)}{(1 - g(r))^2}$$

and

$$r[f''(r)\, f(r) - (f'(r))^2] + f(r)\, f'(r) = \frac{(g'(r) + rg''(r))(1 - g(r) + r(g'(r))^2)}{(1 - g(r))^4},$$

it is clear that (4) and (5) are also satisfied. The proof is complete.

Note that if we let $R = 1$ and $g(r) = r^2$ for $0 < r < 1$, then the function f given by (58), namely $r^2(1 - r^2)^{-1}$, corresponds to the image of the unit hemisphere. Let U be the corresponding (non-circularly symmetric) solution to (1) with $\mathscr{E}$ as in (2) (see Fig. 2). Since $f^{(4)}(0) = 1$, it follows from Theorem 3 that U is not of class C^2. Of course, this result can independently be inferred from the uniqueness results, mentioned in the introduction, due to Deift and Sylvester and Brooks.

Let R be a positive number. Let r_0 and r_1 be such that $0 < r_0 < r_1 < R$. Let $\varphi\colon (0, R) \to [0, 1]$ be a continuous function vanishing on $(0, r_0]$ and equal to 1 on $[r_1, R)$. For each $0 < r < R$, set

$$g(r) = c \int_0^r \varphi(x)(r - x)\, dx,$$

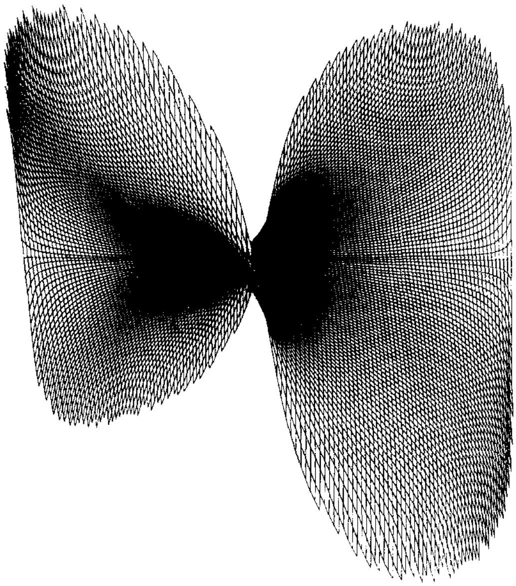

FIG. 2. When viewed from above, under the conditions described in the text, this saddle-like surface will look the same as a hemisphere.

where

$$c = \left[\int_0^R \varphi(x)(R-x)\,dx \right]^{-1}.$$

Clearly, g is of class C^2 and, for each $0 < r < R$,

$$g'(r) = c \int_0^r \varphi(x)\,dx$$

and $g''(r) = c\varphi(r)$. Accordingly, g meets the conditions specified in Theorem 5.

Let f be the function given by (58) and U be the corresponding solution to (1) in which $\mathscr{E}$ is given by (2). Clearly, $\lim_{r \to R} g(r) = 1$ and so $\lim_{r \to R} f(r) = +\infty$. Since g vanishes on $(0, r_0)$, it follows that $f(r) = r^2$ for $0 < r \leqslant r_0$. Thus, by Theorem 4, U is of class C^2 over $D(R)$.

If $r_1 \leqslant r < R$, then

$$g'(r) = c \left[\int_0^{r_1} \varphi(x)\,dx + r - r_1 \right]$$

and so, for $(R+r_1)/2 \leqslant r < R$, $g'(r) \geqslant c(R-r_1)(R+r_1)^{-1} r$. Thus

$$\begin{aligned}\int_0^R \sqrt{f(r)}\, dr &= \int_0^{(R+r_1)/2} \frac{r}{\sqrt{1-g(r)}}\, dr + \int_{(R+r_1)/2}^R \frac{r}{\sqrt{1-g(r)}}\, dr \\ &\leqslant \frac{1}{\sqrt{1-g((R+r_1)/2)}} \int_0^{(R+r_1)/2} r\, dr \\ &\quad + \frac{R+r_1}{c(R-r_1)} \int_{(R+r_1)/2}^R \frac{g'(r)}{\sqrt{1-g(r)}}\, dr \\ &= \frac{(R+r_1)^2}{8\sqrt{1-g((R+r_1)/2)}} + \frac{2(R+r_1)}{c(R-r_1)} \sqrt{1-g((R+r_1)/2)}.\end{aligned}$$

This jointly with [2, Theorem 2] implies that U is bounded.

It is now clear that our goal expressed in the Introduction is achieved: the pair (f, U) provides a desired counterexample to Bruss' assertion.

Acknowledgments

W. Chojnacki expresses gratitude to Flinders University for the opportunity to spend time as a visiting researcher and to Warsaw University for granting leave. The authors are grateful for the assistance of Danny Gibbins in generating and preparing figures.

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