On close-to-scalar one-parameter cosine families

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Abstract

It is shown that if two cosine families with values in a normed algebra with unity, both indexed by t running over all real numbers, of which one consists of the multiples of the unity of the algebra by numbers of the form $\cos at$ for some real a, differ in norm by less than $8/(3\sqrt{3})$ uniformly in t, then these families coincide. For $a \neq 0$, the constant $8/(3\sqrt{3})$ is optimal and cannot be replaced by any larger number.

Keywords: Cosine family, normed algebra, scalar element
2010 MSC: Primary 47D09, 47C05, Secondary 47A30

1. Introduction

The investigations of the present paper have their roots in the classic, albeit not particularly well known, result that if $\mathbf{A}$ is a normed algebra with a unity e and a is an element of $\mathbf{A}$ such that $\sup_{n\in\mathbb{N}} \|a^n - e\| < 1$, then $a = e$. The early version of this result, due to Cox [7], concerned the case of square matrices of a given size. This was later extended to bounded operators on Hilbert space by Nakamura and Yoshida [16], and to an arbitrary normed algebra by Hirschfeld [12] and Wallen [21]. The argument underlying Wallen's contribution was elementary and gave in fact a stronger result, namely that $\|a^n - e\| = o(n)$ and $\liminf_{n\to\infty} n^{-1}\left(\|a - e\| + \|a^2 - e\| + \cdots + \|a^n - e\|\right) < 1$ imply $a = e$.

An immediate consequence of the Cox–Nakamura–Yoshida–Hirschfeld–Wallen theorem is that if $\{S(t)\}_{t\geq 0}$ is a one-parameter semigroup on a Banach space X such that

$$\sup_{t\geq 0} \|S(t) - I_X\| < 1,$$

then $S(t) = I_X$ for each $t \geq 0$; here I_X denotes the identity operator on X. Recently, Bobrowski and Chojnacki [5] established an analogue of this result for one-parameter cosine families: if $a \in \mathbb{R}$ and $C = \{C(t)\}_{t\in\mathbb{R}}$ is a strongly continuous cosine family on a Banach space X such that

$$\sup_{t\in\mathbb{R}} \|C(t) - (\cos at)I_X\| < \frac{1}{2}, \tag{1.1}$$

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then $C(t) = (\cos at)I_X$ for each $t \in \mathbb{R}$. Here, the cosine family $\{(\cos at)I_X\}_{t\in\mathbb{R}}$ against which C is compared is an example of what is termed, following the nomenclature used in [5], a scalar cosine family. Schwenninger and Zwart [19] improved Bobrowski and Chojnacki's result by showing that condition (1.1) can be replaced by the condition

$$\sup_{t\in\mathbb{R}} \|C(t) - (\cos at)I_X\| < 1. \tag{1.2}$$

Chojnacki [6] in turn strengthened Schwenninger and Zwart's result by proving a general theorem that ensures that condition (1.2) alone, without C being necessarily strongly continuous, implies the coincidence of C and $\{(\cos at)I_X\}_{t\in\mathbb{R}}$. Schwenninger and Zwart [20] later showed that in the case $a = 0$, if C is strongly continuous, then the condition

$$\sup_{t\in\mathbb{R}} \|C(t) - I_X\| < 2 \tag{1.3}$$

already suffices to guarantee that $C(t) = I_X$ for each $t \in \mathbb{R}$. This latter result partially generalises the fact that if $\{C(t)\}_{t\in\mathbb{R}}$ is a cosine family on a Banach space X, not necessarily strongly continuous, such that

$$\sup_{t\in\mathbb{R}} \|C(t) - I_X\| < 3/2,$$

then $C(t) = I_X$ for each $t \in \mathbb{R}$. The proof of the above fact is elementary and relies on a straightforward adaptation of the proof of a result of Arendt [4].

The aim of this paper is to generalise the first result of Schwenninger and Zwart (that is, the result related to (1.2)). A consequence of the main result given below is that if $a \in \mathbb{R}$ and $C = \{C(t)\}_{t\in\mathbb{R}}$ is a cosine family on a Banach space X such that

$$\sup_{t\in\mathbb{R}} \|C(t) - (\cos at)I_X\| < \frac{8}{3\sqrt{3}},$$

then $C(t) = (\cos at)I_X$ for each $t \in \mathbb{R}$. Here, the family C need not be assumed strongly continuous. For $a \neq 0$, the constant $8/(3\sqrt{3})$, greater than 1 but less than 2, is optimal and cannot be replaced by any larger number. The main result of the paper will be formulated very much like its consequence just stated, with the exception that the cosine families C and $\{(\cos at)I_X\}_{t\in\mathbb{R}}$ on the Banach space X will be replaced by their counterparts with values in a normed algebra with unity.

2. Preliminaries

Let $\mathbf{A}$ be a normed algebra, real or complex, with a unity e. An element of $\mathbf{A}$ is called *scalar* if it is a scalar multiple of the unity e. A family in $\mathbf{A}$ is termed *scalar* if every member of this family is scalar. A family $\{g(t)\}_{t\in\mathbb{R}}$ in $\mathbf{A}$ is called a one-parameter *group* if

(i) $g(s)g(t) = g(s+t)$ for all $s, t \in \mathbb{R}$ (Cauchy's functional equation, also called the exponential equation),

(ii) $g(0) = e$.

A family $\{c(t)\}_{t\in\mathbb{R}}$ in $\mathbf{A}$ is called a one-parameter *cosine family* if

(i) $2c(s)c(t) = c(s+t) + c(s-t)$ for all $s, t \in \mathbb{R}$ (d'Alembert's functional equation, also called the cosine functional equation),

(ii) $c(0) = e$.

Given a normed linear space X, $\mathscr{L}(X)$ is the normed algebra of all bounded linear operators on X. The identity operator I_X on X is the unity of $\mathscr{L}(X)$.

An $\mathscr{L}(X)$-valued group (cosine family), where X is a normed space, is termed a group (cosine family) on X.

A family $\{x_\lambda\}_{\lambda\in\Lambda}$ with values in a normed space is said to be *bounded* if $\sup_{\lambda\in\Lambda} \|x_\lambda\| < \infty$.

3. A distance result

We first establish an auxiliary result from harmonic analysis.

We recall that a *character* of $\mathbb{R}$ is a function $\chi\colon \mathbb{R} \to \mathbb{C}\setminus\{0\}$ satisfying $\chi(s+t) = \chi(s)\chi(t)$ for any $s, t \in \mathbb{R}$. A character χ of $\mathbb{R}$ is *unitary* if $|\chi(t)| = 1$ for every $t \in \mathbb{R}$.

The result to be proved is as follows.

Theorem 1. *If $a \in \mathbb{R}$ and χ is a character, continuous or not, of $\mathbb{R}$ such that*

$$\sup_{t\in\mathbb{R}} \left| \frac{1}{2}(\chi(t) + \chi(-t)) - \cos at \right| < \frac{8}{3\sqrt{3}},$$

then $\chi(t) = \mathrm{e}^{\mathrm{i}at}$ for all $t \in \mathbb{R}$ or $\chi(t) = \mathrm{e}^{-\mathrm{i}at}$ for all $t \in \mathbb{R}$.

For the proof we shall need two ingredients, and these will be presented in two separate subsections.

3.1. First ingredient

The first ingredient is embodied in the following result.

Theorem 2. *We have*

$$\inf_{\substack{a,b\in\mathbb{R}\\ |a|\neq|b|}} \sup_{t\in\mathbb{R}} |\cos at - \cos bt| = \frac{8}{3\sqrt{3}}.$$

As a first step towards the proof, for any pair $a, b \in \mathbb{R}$, we define a function $f_{a,b}\colon \mathbb{R} \to \mathbb{R}$ by

$$f_{a,b}(t) = \cos at - \cos bt \quad (t \in \mathbb{R}).$$

If we let

$$K := \inf_{\substack{a,b\in\mathbb{R}\\ |a|\neq|b|}} \|f_{a,b}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the standard supremum norm, then Theorem 2 can be formulated as saying that $K = 8/(3\sqrt{3})$.

Given that $\|f_{a,b}\|_\infty \le 2$ for any $a, b \in \mathbb{R}$, we have $K \le 2$. Furthermore, since $f_{a,b} = f_{-a,b} = f_{a,-b} = f_{-a,-b}$ and $|f_{a,b}| = |f_{b,a}|$ for any $a, b \in \mathbb{R}$, it follows that

$$K = \inf_{a>b\ge 0} \|f_{a,b}\|_\infty.$$

Now, $\|f_{a,0}\|_\infty = 2$ for any $a \neq 0$, and if $b \neq 0$, then $\|f_{a,b}\|_\infty = \|f_{a/b,1}\|_\infty$ for any $a \in \mathbb{R}$. This together with the previous observations implies that

$$K = \inf_{c>1} \|g_c\|_\infty,$$

where, for any $c \in \mathbb{R}$, g_c is short for $f_{c,1}$.

Lemma 1. *We have*

$$\|g_3\|_\infty = \frac{8}{3\sqrt{3}}.$$

Proof. For each $t \in \mathbb{R}$,

$$g_3(t) = -2\sin 2t \sin t = -4\sin^2 t \cos t$$

and

$$g_3'(t) = -4(2 - 3\sin^2 t)\sin t.$$

Being a bounded periodic function, g_3 attains its extrema at points t satisfying $\sin t = 0$ or $\sin^2 t = 2/3$. As $g_3(t) = 0$ if $\sin t = 0$, g_3 attains in fact its extrema at points t satisfying $\sin^2 t = 2/3$. Now, if t is any such point, then

$$|g_3(t)| = 4\sin^2 t|\cos t| = 4\sin^2 t\sqrt{1 - \sin^2 t} = 4 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{8}{3\sqrt{3}}.$$

This establishes the lemma.

Lemma 2. *Let $c = p/q > 1$ be a rational number, with p and q relatively prime positive integers. Then*

(i) $\|g_c\|_\infty = 2$ *if either p or q is even;*

(ii) $\|g_c\|_\infty \geq 1 + \cos(\pi/p)$ *if both p and q are odd.*

Proof. To prove assertion (i), note first that, being relatively prime, p and q cannot be both even. Thus if either p or q is even, then p and q have different parity. This implies that, with $t = q\pi$, we have

$$g_c(t) = \cos p\pi - \cos q\pi = (-1)^p - (-1)^q,$$

so that $|g_c(t)| = 2$ and further $\|g_c\|_\infty = 2$.

To prove assertion (ii), assume that both p and q are odd. Since p and q are relatively prime, there exist integers m and n such that $pm + qn = 1$. Setting $t = qn\pi/p$, we obtain

$$g_c(t) = \cos n\pi - \cos\frac{qn\pi}{p} = (-1)^n - \cos\left(\frac{1}{p} - m\right)\pi.$$

The oddness of p and q implies that m and n are of different parity. Thus, if m is odd, then n is even and we have

$$g_c(t) = 1 + \cos\frac{\pi}{p};$$

and if m is even, then n is odd and we have

$$g_c(t) = -1 - \cos\frac{\pi}{p}.$$

In both cases, $\|g_c\|_\infty \geq |g_c(t)| = 1 + \cos(\pi/p)$.

Proposition 1. *We have*

$$\inf_{\substack{c\in\mathbb{Q}\\ c>1}} \|g_c\|_\infty = \min_{\substack{c\in\mathbb{Q}\\ c>1}} \|g_c\|_\infty = \|g_3\|_\infty = \frac{8}{3\sqrt{3}}.$$

Proof. In view of Lemma 1, it suffices to show that $\|g_c\|_\infty > 8/(3\sqrt{3})$ whenever c is a rational number different from 3. Let $c = p/q > 1$, with p and q relatively prime positive integers, be different from 3. If either p or q is even, then, by assertion (i) of Lemma 2, we have $\|g_c\|_\infty = 2$ and a fortiori $\|g_c\|_\infty > 8/(3\sqrt{3})$. If both p and q are odd, then, since $c > 1$ and $c \neq 3$, we have $p \geq 5$, and further, by assertion (ii) of Lemma 2,

$$\|g_c\|_\infty \geq 1 + \cos\frac{\pi}{p} > 1 + \cos\frac{\pi}{4} = 1 + \frac{\sqrt{2}}{2} > \frac{8}{3\sqrt{3}}.$$

Since p and q cannot be simultaneously even, the lemma is established.

Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$.

Proposition 2. *We have*

$$\inf_{\substack{c\in\mathbb{R}\setminus\mathbb{Q}\\ c>1}} \|g_c\|_\infty = 2.$$

Proof. Let $c > 1$ be an irrational number. As is well known, the image of the mapping $\mathbb{R} \ni t \mapsto (\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}ct}) \in \mathbb{T}^2$ is dense in $\mathbb{T}^2$ (see e.g. [1, Example 4.4.10] or [3, Example 2.3.10]). In particular, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers such that $\lim_{n\to\infty} \mathrm{e}^{\mathrm{i}t_n} = -1$ and $\lim_{n\to\infty} \mathrm{e}^{\mathrm{i}ct_n} = 1$. Now, $\lim_{n\to\infty} g_c(t_n) = 2$, and this implies that $\|g_c\|_\infty = 2$.

At this stage, it is clear that Theorem 2 is an immediate consequence of Propositions 1 and 2. We have the following direct corollary to Theorem 2.

Corollary 1. *If $a, b \in \mathbb{R}$ are such that*

$$\sup_{t\in\mathbb{R}} |\cos at - \cos bt| < \frac{8}{3\sqrt{3}},$$

then $a = b$ or $a = -b$, and in either case $\cos at = \cos bt$ for all $t \in \mathbb{R}$.

3.2. Second ingredient

We start with an auxiliary result.

Proposition 3. *If χ is a discontinuous unitary character of $\mathbb{R}$, then every element of $\mathbb{T}$ is a cluster point of the net $\{\chi(t)\}_{t\to 0}$.*

Proof. Let χ be a discontinuous character of $\mathbb{R}$, and let C be the set of all cluster points of the net $\{\chi(t)\}_{t\to 0}$, which is the same as the set of all limits $\lim_{n\to\infty} \chi(t_n)$ of converging sequences $\{\chi(t_n)\}_{n\in\mathbb{N}}$, where $\{t_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers tending to zero. If c and d are in C and $c = \lim_{n\to\infty} \chi(s_n)$ and $d = \lim_{n\to\infty} \chi(t_n)$, where $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ are two sequences of real numbers converging to zero, then $c^{-1} = \lim_{n\to\infty} \chi(-s_n)$ and $cd = \lim_{n\to\infty} \chi(s_n)\chi(t_n) = \lim_{n\to\infty} \chi(s_n + t_n)$, so both c^{-1} and cd belong to C. In other words, C is a group under multiplication. Clearly, C is

also a closed subset of $\mathbb{T}$. Thus C is a closed subgroup of the circle group $\mathbb{T}$, and as such it is either finite or all of $\mathbb{T}$ (see e.g. [15, Section 2, Corollary 3]).

Suppose that the first possibility holds. Then there exists $k \in \mathbb{N}$ such that $c^k = 1$ for all $c \in C$. Fix $c \in C$ arbitrarily and let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers converging to zero such that $c = \lim_{n\to\infty} \chi(t_n)$. By the compactness of $\mathbb{T}$, there exists a sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers diverging to infinity such that $\{\chi(t_{n_m}/k)\}_{m\in\mathbb{N}}$ converges. Consequently, $d = \lim_{m\to\infty} \chi(t_{n_m}/k)$ belongs to C. Clearly, $c = d^k$. On the other hand, d, as any other member of C, obeys $d^k = 1$, so $c = 1$. Thus $C = \{1\}$, and this implies that χ is continuous at 0, and hence everywhere on $\mathbb{R}$, contrary to the assumption. It follows that the other possibility concerning the form of C holds, namely $C = \mathbb{T}$.

We are now ready to present the second ingredient needed for the proof of Theorem 1.

Theorem 3. *If $a \in \mathbb{R}$ and χ is a discontinuous unitary character of $\mathbb{R}$, then*

$$\limsup_{t\to 0} \left| \frac{1}{2}(\chi(t) + \chi(-t)) - \cos at \right| = 2.$$

Proof. Let $a \in \mathbb{R}$ and let χ be a discontinuous unitary character of $\mathbb{R}$. By Proposition 3, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers tending to zero such that $\lim_{n\to\infty} \chi(t_n) = -1$. Now, $\lim_{n\to\infty} \chi(-t_n) = \lim_{n\to\infty} (\chi(t_n))^{-1} = -1$ and $\lim_{n\to\infty} \cos at_n = 1$. Hence

$$\lim_{n\to\infty} \left| \frac{1}{2}(\chi(t_n) + \chi(-t_n)) - \cos at_n \right| = 2$$

and further

$$\limsup_{t\to 0} \left| \frac{1}{2}(\chi(t) + \chi(-t)) - \cos at \right| \geq 2.$$

But $|(\chi(t) + \chi(-t))/2 - \cos at| \leq 2$ for all $t \in \mathbb{R}$, so in fact

$$\limsup_{t\to 0} \left| \frac{1}{2}(\chi(t) + \chi(-t)) - \cos at \right| = 2,$$

as was to be proved.

As a side remark, we point out that Theorem 3 can be reformulated as follows.

Theorem 4. *If $a \in \mathbb{R}$ and $\{c(t)\}_{t\in\mathbb{R}}$ is a real-valued bounded discontinuous cosine family, then*

$$\sup_{t\in\mathbb{R}} |c(t) - \cos at| = 2.$$

To see that Theorems 3 and 4 are in fact equivalent, we first recall a few facts. The complex-valued cosine families are precisely the families $\{c(t)\}_{t\in\mathbb{R}}$ of the form

$$c(t) = \frac{1}{2}(\chi(t) + \chi(-t)) \quad (t \in \mathbb{R}), \tag{3.1}$$

where χ is a possibly non-unitary character of $\mathbb{R}$ [14, Theorem 2]. The representation (3.1) is essentially unique in that if χ and γ are two characters of $\mathbb{R}$ satisfying $\chi(t) + \chi(-t) = \gamma(t) + \gamma(-t)$ for all $t \in \mathbb{R}$, then either $\gamma = \chi$ or $\gamma = \check{\chi}$, where $\check{\chi}$ is the character of $\mathbb{R}$ defined by $\check{\chi}(t) = \chi(-t)$ for

all $t \in \mathbb{R}$ [14, Theorem 3]. A complex-valued cosine family $\{c(t)\}_{t\in\mathbb{R}}$ is continuous (at a single point, or, equivalently, everywhere) if and only if the corresponding character χ is continuous [14, Theorem 1]. A complex-valued cosine family $\{c(t)\}_{t\in\mathbb{R}}$ is bounded if and only if χ is bounded, which happens precisely when χ is unitary (see e.g. [5, proof of Theorem 10]). It follows that a bounded complex-valued cosine family is in fact real-valued.

In light of the preceding remarks, if $c = \{c(t)\}_{t\in\mathbb{R}}$ is a real-valued bounded discontinuous cosine family, then c has a representation as per (3.1) with χ being unitary and discontinuous. Conversely, if χ is a discontinuous unitary character of $\mathbb{R}$, then the cosine family given by (3.1) is bounded and discontinuous. Now the equivalence of Theorems 3 and 4 is clear.

3.3. Proof of Theorem 1

We are now in position to present the proof of Theorem 1.

Proof of Theorem 1. Let $a \in \mathbb{R}$ and let χ be a character of $\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \left| \frac{1}{2}(\chi(t) + \chi(-t)) - \cos at \right| < \frac{8}{3\sqrt{3}}. \tag{3.2}$$

Then, clearly, the cosine family $\{(\chi(t)+\chi(-t))/2\}_{t\in\mathbb{R}}$ is bounded, and, in accordance with remarks made earlier, χ is unitary. Since $8/(3\sqrt{3}) < 2$, it follows from Theorem 3 that χ is continuous. Consequently, there exists $b \in \mathbb{R}$ such that $\chi(t) = \mathrm{e}^{ibt}$ for all $t \in \mathbb{R}$. Now, as $(\chi(t) + \chi(-t))/2 = \cos bt$ for all $t \in \mathbb{R}$, Corollary 1 jointly with (3.2) implies that $a = b$ or $a = -b$, yielding the assertion of the theorem.

4. A generalisation of a theorem of Gelfand

We next establish an auxiliary result from Banach algebra theory.

For an element a of a complex normed algebra $\mathbf{A}$, we denote the resolvent set and spectrum of a by $\rho(a)$ and $\sigma(a)$, respectively; when more specificity is required, the notation $\rho_{\mathbf{A}}(a)$ and $\sigma_{\mathbf{A}}(a)$ is used instead.

We recall that an invertible element a of a normed algebra with unity is termed *doubly power bounded* if $\sup_{n\in\mathbb{Z}} \|a^n\| < \infty$. From the spectral radius formula it follows that the spectrum of a doubly power bounded element of a complex Banach algebra with unity is contained in $\mathbb{T}$.

A result of significance for us concerning doubly power bounded elements is the following.

Theorem 5 (Gelfand [9]). *Let* $\mathbf{A}$ *be a complex Banach algebra with a unity* e *and let* a *be a doubly power bounded element of* $\mathbf{A}$. *If* $\sigma(a) = \{1\}$, *then* $a = e$.

Gelfand's theorem can be proved in a number of different ways (see e.g. [2, Theorem 1.1], [9], [17, Corollary 4.2]). The result has various generalisations, of which one is due to Hille [10] (see also [11, Theorem 4.10.1]) and is usually referred to as the Gelfand–Hille theorem; it states that if a is an element of a complex Banach algebra with a unit e such that $\sigma(a) = \{1\}$, then $(a - e)^r = 0$ for some $r \in \mathbb{N}$ if and only if $\|a^n\| = O(n^{r-1})$, or $\|a^n\| = o(n^r)$, as $|n| \to \infty$. For an informative account of various developments related to the Gelfand–Hille theorem, see [22]; and for modern generalisations of this theorem, see [8] and the references therein.

Below we establish a generalisation of Gelfand's theorem tailored specifically to the needs of the current exposition.

Theorem 6. *Let* $\mathbf{A}$ *be a complex Banach algebra with a unity* e*, and let* a *be a doubly power bounded element of* $\mathbf{A}$ *with a finite spectrum* $\sigma(a) = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{T}$*,* $\lambda_k \neq \lambda_l$ *for* $k \neq l$*. Then there exist idempotents* $p_1, \dots, p_n$ *in* $\mathbf{A}$ *such that* $\sum_{k=1}^n p_k = e$*,* $p_k p_l = 0$ *for* $k \neq l$*, and* $a = \sum_{k=1}^n \lambda_k p_k$*.*

Proof. Let $\mathbf{B}$ be the Banach algebra generated by all elements of the form $(\lambda e - a)^{-1}$, $\lambda \in \rho_{\mathbf{A}}(a)$. Then $\mathbf{B}$ is a commutative algebra, and, since $\lim_{\lambda\to\infty} \lambda(\lambda e - a)^{-1} = e$ and $\lim_{\lambda\to\infty}[\lambda^2(\lambda e - a)^{-1} - \lambda e] = a$, $\mathbf{B}$ contains e and a.

The single-variable analytic functional calculus guarantees the existence of idempotents $p_1, \dots, p_n$ in $\mathbf{A}$ with the following properties:

(i) $\sum_{k=1}^n p_k = e$, $p_k \neq 0$ for $1 \leq k \leq n$, and $p_k p_l = 0$ for $1 \leq k, l \leq n$ with $k \neq l$;

(ii) $\sigma_{\mathbf{A}}(ap_k) \subset \{0, \lambda_k\}$ for $1 \leq k \leq n$;

(iii) each p_k is contained in the closed linear span of all elements of the form $(\lambda e - a)^{-1}$, $\lambda \in \rho_{\mathbf{A}}(a)$.

(Cf. [13, Theorem 3.2.9] and [18, Proposition 3.4.1].) Note that condition (iii) implies that all the p_k's belong to $\mathbf{B}$.

Given $1 \leq k \leq n$, let

$$\mathbf{I}_k = \{c \in \mathbf{B} \mid c = bp_k,\ b \in \mathbf{B}\}.$$

$\mathbf{I}_k$ is a closed ideal of $\mathbf{B}$ and hence a commutative Banach algebra, and p_k is the unity of $\mathbf{I}_k$. For any $c \in \mathbf{I}_k$, we have $\sigma_{\mathbf{I}_k}(c) \subset \sigma_{\mathbf{A}}(c)$. Indeed, if $\lambda \in \rho_{\mathbf{A}}(c)$, then

$$p_k = (\lambda e - c)^{-1}(\lambda e - c)p_k = (\lambda e - c)^{-1} p_k(\lambda p_k - c),$$

so $(\lambda e - c)^{-1} p_k$ is the inverse of $\lambda p_k - c$ in $\mathbf{I}_k$ and hence $\lambda \in \rho_{\mathbf{I}_k}(c)$; in other words, $\rho_{\mathbf{A}}(c) \subset \rho_{\mathbf{I}_k}(c)$ which is equivalent to $\sigma_{\mathbf{I}_k}(c) \subset \sigma_{\mathbf{A}}(c)$. In particular,

$$\sigma_{\mathbf{I}_k}(ap_k) \subset \sigma_{\mathbf{A}}(ap_k) = \{\lambda_k, 0\}. \tag{4.1}$$

As $a^{-1}p_k a p_k = p_k$, $a^{-1}p_k$ is the inverse of ap_k in $\mathbf{I}_k$. Therefore 0 is outside $\sigma_{\mathbf{I}_k}(ap_k)$, and relation (4.1) reduces to $\sigma_{\mathbf{I}_k}(ap_k) \subset \{\lambda_k\}$. As $\sigma_{\mathbf{I}_k}(ap_k)$ is non-void, we in fact have $\sigma_{\mathbf{I}_k}(ap_k) = \{\lambda_k\}$. Consider now the element $c_k = \overline{\lambda_k} a p_k$. Clearly, $\sigma_{\mathbf{I}_k}(c_k) = \{1\}$. The inverse of c_k in $\mathbf{I}_k$ equals $\lambda_k a^{-1} p_k$. Moreover,

$$\sup_{n\in\mathbb{Z}} \|c_k^n\| \leq \|p_k\| \sup_{n\in\mathbb{Z}} \|a^n\|,$$

so c_k is doubly power bounded. By Gelfand's theorem, c_k coincides with the unity of $\mathbf{I}_k$, that is, $c_k = p_k$. Hence $ap_k = \lambda_k p_k$.

Finally, we have

$$a = \sum_{k=1}^n ap_k = \sum_{k=1}^n \lambda_k p_k,$$

which is exactly what is needed to complete the proof.

5. Main result

This section contains the main result of the paper, which is as follows.

Theorem 7. *Let* $\mathbf{A}$ *be a normed algebra with a unity* e*, let* $a \in \mathbb{R}$*, and let* $\{c(t)\}_{t\in\mathbb{R}}$ *be an* $\mathbf{A}$*-valued cosine family such that*

$$\sup_{t\in\mathbb{R}} \|c(t) - (\cos at)e\| < \frac{8}{3\sqrt{3}}.$$

Then $c(t) = (\cos at)e$ *for all* $t \in \mathbb{R}$.

Proof. Let $l^\infty(\mathbb{R}, \mathbf{A})$ be the space of all bounded functions from $\mathbb{R}$ to $\mathbf{A}$, endowed with the norm

$$\|x\|_\infty = \sup_{s\in\mathbb{R}} \|x(s)\| \quad (x \in l^\infty(\mathbb{R}, \mathbf{A})).$$

For an element a of $\mathbf{A}$, let $\underline{a}$ denote the constant function on $\mathbb{R}$ with value a. For each $t \in \mathbb{R}$, we define a linear operator $C(t)$ on $l^\infty(\mathbb{R}, \mathbf{A})$ by

$$(C(t)x)(s) = c(t)x(s) \quad (x \in l^\infty(\mathbb{R}, \mathbf{A}),\ s \in \mathbb{R}).$$

Clearly, $C(t)$ is bounded, with $\|C(t)\| \le \|c(t)\|$. Since $C(t)\underline{e} = c(t)$ and $\|\underline{e}\|_\infty = 1$, we see that in fact $\|C(t)\| = \|c(t)\|$. It is plain that $\{C(t)\}_{t\in\mathbb{R}}$ is a cosine family on $l^\infty(\mathbb{R}, \mathbf{A})$. For each $t \in \mathbb{R}$, we have

$$((C(t) - (\cos at)I_{l^\infty(\mathbb{R},\mathbf{A})})x)(s) = (c(t) - (\cos at)e)x(s)$$

for any $x \in l^\infty(\mathbb{R}, \mathbf{A})$ and any $s \in \mathbb{R}$, and this implies, as above, that

$$\|C(t) - (\cos at)I_{l^\infty(\mathbb{R},\mathbf{A})}\| = \|c(t) - (\cos at)e\|.$$

Let $0 < \delta < 8/(3\sqrt{3})$ be such that $\|c(t) - \cos(at)e\| \le \delta$ for all $t \in \mathbb{R}$. Then, clearly,

$$\|C(t) - (\cos at)I_{l^\infty(\mathbb{R},\mathbf{A})}\| \le \delta \tag{5.1}$$

for all $t \in \mathbb{R}$.

Given $t \in \mathbb{R}$, let T_t denote the operator of *translation* by t on $l^\infty(\mathbb{R}, \mathbf{A})$ defined by

$$(T_t x)(s) = x(s + t) \quad (x \in l^\infty(\mathbb{R}, \mathbf{A}),\ s \in \mathbb{R}).$$

Clearly, T_t is a surjective linear isometry, with inverse T_{-t}.

As $\|c(t)\| \le \|c(t) - (\cos at)e\| + |\cos at|\|e\| \le \delta + 1$ for all $t \in \mathbb{R}$, $c = \{c(t)\}_{t\in\mathbb{R}}$ is bounded—that is, c is a member of $l^\infty(\mathbb{R}, \mathbf{A})$. Let Z be the linear space of all functions z in $l^\infty(\mathbb{R}, \mathbf{A})$ of the form

$$z = \sum_{k=1}^{n} \alpha_k T_{s_k} c,$$

where $\alpha_k \in \mathbb{C}$ and $s_k \in \mathbb{R}$ for $k = 1, \ldots, n$. It is clear that Z is invariant under each T_t, $t \in \mathbb{R}$. Given $S \in \mathscr{L}(X)$, where X is a linear space, and $Y \subset X$ such that $S(Y) \subset Y$, we denote by $S|_Y$ the restriction of S to Y. For each $t \in \mathbb{R}$, let

$$\tilde{T}_t = T_t|_Z.$$

The family $\{\tilde{T}_t\}_{t\in\mathbb{R}}$ is a one-parameter group with values in the normed algebra $\mathscr{L}(Z)$. Moreover, since $\|\tilde{T}_t\| \le 1$, $\|\tilde{T}_{-t}\| \le 1$, and $1 = \|I_Z\| \le \|\tilde{T}_t\|\,\|\tilde{T}_{-t}\|$, we see that $\|\tilde{T}_t\| = 1$ for each $t \in \mathbb{R}$.

For each $t \in \mathbb{R}$, if z is a member of Z, $z = \sum_{k=1}^{n} \alpha_k T_{s_k} c$, where $\alpha_k \in \mathbb{C}$ and $s_k \in \mathbb{R}$ for $k = 1, \dots, n$, and if $s \in \mathbb{R}$, then

$$\begin{aligned}(C(t)z)(s) &= c(t)\sum_{k=1}^{n} \alpha_k c(s+s_k)\\ &= \frac{1}{2}\left(\sum_{k=1}^{n} \alpha_k c(s+s_k+t) + \sum_{k=1}^{n} \alpha_k c(s+s_k-t)\right)\\ &= \frac{1}{2}(T_t z + T_{-t} z)(s).\end{aligned}$$

Thus Z is an invariant subspace for $C(t)$ and we have

$$C(t)|_Z = \frac{1}{2}(\tilde{T}_t + \tilde{T}_{-t}).$$

Let $\mathbf{B}_0$ be the subalgebra of $\mathscr{L}(Z)$ generated by the $\tilde{T}_t$'s. Obviously, $\mathbf{B}_0$ is a commutative normed algebra with unity, the unity element being the identity operator I_Z. Let $\tilde{C} = \{\tilde{C}(t)\}_{t\in\mathbb{R}}$ be the $\mathbf{B}_0$-valued cosine family defined by

$$\tilde{C}(t) = \frac{1}{2}(\tilde{T}_t + \tilde{T}_{-t}) \quad (t \in \mathbb{R}).$$

Then, for each $t \in \mathbb{R}$,

$$\tilde{C}(t) = C(t)|_Z,$$

and further, on account of (5.1),

$$\|\tilde{C}(t) - (\cos at)I_Z\| \le \delta.$$

Let $\mathbf{B}$ denote the completion of $\mathbf{B}_0$, complexified if $\mathbf{B}_0$ is real. Clearly, $\mathbf{B}$ is a Banach algebra with unity, the unity element being again I_Z. Let $\Delta(\mathbf{B})$ denote the set of all complex-valued homomorphisms on $\mathbf{B}$, and let $\phi \in \Delta(\mathbf{B})$. Then

$$\phi(\tilde{T}_{s+t}) = \phi(\tilde{T}_s\tilde{T}_t) = \phi(\tilde{T}_s)\phi(\tilde{T}_t)$$

for any $s, t \in \mathbb{R}$, so the function $\mathbb{R} \ni t \mapsto \phi(\tilde{T}_t) \in \mathbb{C}$ is a character of $\mathbb{R}$. Moreover, for each $t \in \mathbb{R}$,

$$\begin{aligned}\left|\frac{1}{2}\left(\phi(\tilde{T}_t) + \phi(\tilde{T}_{-t})\right) - \cos at\right| &= \left|\phi\left(\frac{1}{2}(\tilde{T}_t + \tilde{T}_{-t}) - (\cos at)I_Z\right)\right|\\ &\le \|\phi\|\,\|\tilde{C}(t) - (\cos at)I_Z\| \le \delta,\end{aligned}$$

where we used the fact that $\|\phi\| = 1$ (see e.g. [13, Lemma 2.1.5]). Now, Theorem 1 guarantees that $\phi(\tilde{T}_t) = \mathrm{e}^{\mathrm{i}at}$ for all $t \in \mathbb{R}$ or $\phi(\tilde{T}_t) = \mathrm{e}^{-\mathrm{i}at}$ for all $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$ arbitrarily. Then

$$\sigma_{\mathbf{B}}(\tilde{T}_t) = \{\phi(\tilde{T}_t) \mid \phi \in \Delta(\mathbf{B})\} \subset \{\mathrm{e}^{\mathrm{i}at}, \mathrm{e}^{-\mathrm{i}at}\},$$

where the equality on the left hand side stems from the well-known fact that $\sigma_{\mathbf{C}}(c) = \{\phi(c) \mid \phi \in \Delta(\mathbf{C})\}$ for any element c of a complex commutative Banach algebra $\mathbf{C}$ with unity (see e.g. [13,

Theorem 2.2.5]). Since $\tilde{T}_t$ is doubly power bounded (recall that $\|\tilde{T}_t\| = 1$ and $\|\tilde{T}_t^{-1}\| = \|\tilde{T}_{-t}\| = 1$), it follows from Theorem 6 that there exists an idempotent P_t in $\mathbf{B}$ such that

$$\tilde{T}_t = \mathrm{e}^{\mathrm{i}at}P_t + \mathrm{e}^{-\mathrm{i}at}(I_Z - P_t).$$

Note in passing that if $\sigma_{\mathbf{B}}(\tilde{T}_t)$ consists of a single element, then P_t is either zero or I_Z. Since

$$\begin{aligned}(\mathrm{e}^{\mathrm{i}at}P_t + \mathrm{e}^{-\mathrm{i}at}(I_Z - P_t))&(\mathrm{e}^{-\mathrm{i}at}P_t + \mathrm{e}^{\mathrm{i}at}(I_Z - P_t))\\ &= P_t^2 + \mathrm{e}^{-2a\mathrm{i}t}(I_Z - P_t)P_t + \mathrm{e}^{2a\mathrm{i}t}P_t(I_Z - P_t) + (I_Z - P_t)^2\\ &= P_t^2 + (I_Z - P_t)^2 = P_t + I_Z - P_t = I_Z,\end{aligned}$$

it follows that $\mathrm{e}^{-\mathrm{i}at}P_t + \mathrm{e}^{\mathrm{i}at}(I_Z - P_t)$ is the inverse of $\mathrm{e}^{\mathrm{i}at}P_t + \mathrm{e}^{-\mathrm{i}at}(I_Z - P_t)$. We thus have

$$\tilde{T}_{-t} = \mathrm{e}^{-\mathrm{i}at}P_t + \mathrm{e}^{\mathrm{i}at}(I_Z - P_t).$$

Consequently,

$$\tilde{C}(t) = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}at}P_t + \mathrm{e}^{-\mathrm{i}at}(I_Z - P_t) + \mathrm{e}^{-\mathrm{i}at}P_t + \mathrm{e}^{\mathrm{i}at}(I_Z - P_t)\right) = (\cos at)I_Z.$$

In particular, $\tilde{C}(t)c = (\cos at)c$. But

$$(\tilde{C}(t)c)(0) = \frac{1}{2}(c(t) + c(-t)) = c(t)$$

and

$$((\cos at)c)(0) = (\cos at)c(0) = (\cos at)e.$$

Therefore $c(t) = (\cos at)e$. Since t was arbitrary, the theorem is proved.

We conclude the paper with two remarks.

Remark 1. For $a \neq 0$, the constant $8/(3\sqrt{3})$ in Theorem 7 is optimal and cannot be replaced by any larger number. This follows from Lemma 1 and the observation that

$$\sup_{t\in\mathbb{R}} |\cos 3at - \cos at| = \sup_{t\in\mathbb{R}} |\cos 3t - \cos t|$$

whenever $a \in \mathbb{R} \setminus \{0\}$.

Remark 2. An immediate consequence of Theorem 7 is the fact that in the set of all bounded cosine families with values in a given normed algebra with a unity e, endowed with the metric of the uniform convergence corresponding to the norm of the algebra, any scalar cosine family of the form $\{(\cos at)e\}_{t\in\mathbb{R}}$, $a \in \mathbb{R}$, is an isolated point. A number of results related to this fact can be found in [5].

Note added in proof

Since submitting this work for publication, we have learnt that similar results were obtained independently, but somewhat later, by J. Esterle [arXiv:1502.00150].

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