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ON THE BERGMAN METRIC TENSOR

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SUMMARY. Two necessary and sufficient conditions are given for the Bergman metric tensor to be positive definite. One of them is used to prove that the field of the Bergman tensors over an open subset of $\mathbb{C}$ with nontrivial Bergman space is everywhere positive definite.

Let Ω be an open subset of $\mathbb{C}^n$ $(n \in \mathbb{N})$. Let $L^2H(\Omega)$ be the Bergman space of Ω, that is, the Hilbert space of all Lebesgue square integrable holomorphic functions on Ω equipped with the standard product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, let $|\alpha| = \alpha_1 + \dots \alpha_n$. As is well known, given $w \in \Omega$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$, the linear functional

$$L^2H(\Omega) \ni f \mapsto \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(w) \in \mathbb{C}$$

is bounded. Hence, by the Riesz theorem, for each $w \in \Omega$ and each $\alpha \in (\mathbb{N} \cup \{0\})^n$ there exists a unique element $\partial_{\bar{z}_1^{\alpha_1} \dots \bar{z}_n^{\alpha_n}} \chi_w$ of $L^2H(\Omega)$ such that

$$\frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(w) = (f, \partial_{\bar{z}_1^{\alpha_1} \dots \bar{z}_n^{\alpha_n}} \chi_w) \quad \big(f \in L^2H(\Omega)\big). \tag{1}$$

Let K_Ω be the Bergman function of Ω given by

$$K_\Omega(w, z) = (\chi_z, \chi_w) \quad (w, z \in \Omega).$$

Then, by (1), for any $w, z \in \Omega$ and any $\alpha \in (\mathbb{N} \cup \{0\})^n$,

$$\begin{aligned} \frac{\partial^{|\alpha|} K_\Omega}{\partial \bar{z}_1^{\alpha_1} \dots \partial \bar{z}_n^{\alpha_n}}(w, z) &= \overline{\frac{\partial^{|\alpha|} \overline{K_\Omega}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}(w, z)} = \overline{(\chi_w, \partial_{\bar{z}_1^{\alpha_1} \dots \bar{z}_n^{\alpha_n}} \chi_z)} \\ &= \partial_{\bar{z}_1^{\alpha_1} \dots \bar{z}_n^{\alpha_n}} \chi_z(w), \end{aligned}$$

and hence, for any $w, z \in \Omega$ and $\alpha, \beta \in (\mathbb{N} \cup \{0\})^n$,

$$\frac{\partial^{|\alpha|+|\beta|} K_\Omega}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \dots \partial \bar{z}_n^{\beta_n}}(w, z) = (\partial_{\bar{z}_1^{\beta_1} \dots \bar{z}_n^{\beta_n}} \chi_z, \partial_{\bar{z}_1^{\alpha_1} \dots \bar{z}_n^{\alpha_n}} \chi_w). \tag{2}$$

Suppose that $K_\Omega(w, w) > 0$ (or, equivalently, $\chi_w \neq 0$) for $w \in \Omega$. Then one may define the Bergman metric tensor g_Ω at ω by setting

$$g_\Omega(w) = \sum_{i,j=1}^{n} \frac{\partial^2 \log K_\Omega}{\partial z_i \partial \bar{z}_j}(w, w) dz_i d\bar{z}_j.$$

We have the following

Theorem 1. *Let Ω be an open subset of $\mathbb{C}^n$ $(n \in \mathbb{N})$ such that $L^2H(\Omega) \neq \{0\}$. Suppose that $K_\Omega(w, w) > 0$ for $w \in \Omega$. Then g_Ω is positive definite if and only if the $\partial_{\bar{z}_i} \chi_w$ $(1 \leqslant i \leqslant n)$ and χ_w are linearly independent.*

Proof. For each $1 \leqslant i \leqslant n$ and each $1 \leqslant j \leqslant n$, we have

$$\frac{\partial^2 \log K_\Omega}{\partial z_i \partial \bar{z}_j}(w,w) = (K_\Omega(w,w))^{-2}\left[\frac{\partial^2 K_\Omega}{\partial z_i \partial \bar{z}_j}(w,w)K_\Omega(w,w) - \frac{\partial K_\Omega}{\partial z_i}(w,w)\frac{\partial K_\Omega}{\partial \bar{z}_j}(w,w)\right].$$

Hence, by (2), for any $\alpha_1, \dots, \alpha_n \in \mathbb{C}$,

$$\frac{\partial^2 \log K_\Omega}{\partial z_i \partial \bar{z}_j}(w,w)\alpha_i\overline{\alpha}_j = (K_\Omega(w,w))^{-2}\left[\Big\|\sum_{i=1}^n \overline{\alpha}_i \partial_{\bar{z}_i \chi_w}\Big\|^2 \|\chi_w\|^2 - \Big|\Big(\sum_{i=1}^n \overline{\alpha}_i \partial_{\bar{z}_i}\chi_w, \chi_w\Big)\Big|^2\right]. \tag{3}$$

Suppose that the $\partial_{\bar{z}_i}\chi_w$ $(1 \leqslant i \leqslant n)$ and χ_w are linearly independent. Suppose, moreover, that

$$\frac{\partial^2 \log K_\Omega}{\partial z_i \partial \bar{z}_j}(w,w)\alpha_i\overline{\alpha}_j = 0$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then, in view of (3) and the fact that equality in the Cauchy–Schwarz inequality occurs precisely when the vectors entering the inequality are linearly dependent, there exists $\beta \in \mathbb{C}$ such that

$$\sum_{i,j=1}^n \overline{\alpha}_i \partial_{\bar{z}_i}\chi_w = \beta\chi_w. \tag{4}$$

Hence $\alpha_1 = \dots = \alpha_n = \beta = 0$, which shows that $g_\Omega(w)$ is positive definite.

Conversely, if $g_\Omega(w)$ is positive definite and (4) holds for some $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{C}$, then, by (3), $\alpha_1 = \dots = \alpha_n = 0$, and further, by (4), $\beta = 0$, showing that the $\partial_{\bar{z}_i}\chi_w$ $(1 \leqslant i \leqslant n)$ and χ_w are linearly independent. □

Note that an easy argument to prove the non-negative definiteness of the Bergman tensor, whenever the latter is defined, is to combine (3) with the Cauchy–Schwarz inequality.

Note also that Theorem 1 can be used to re-derive in a simple way the classical result stating that the field of the Bergman tensors over an open bounded subset Ω of $\mathbb{C}^n$ $(n \in \mathbb{N})$ is everywhere positive definite. Indeed, it is well known that $K_\Omega(w,w) > 0$ for all $w \in \Omega$. Moreover, if (4) holds for some $w \in \Omega$ and some $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{C}$, then by (1) for any $h \in L^2H(\Omega)$

$$\sum_{i=1}^n \overline{\alpha}_i \overline{\frac{\partial h}{\partial z_i}(w)} = \Big(\sum_{i=1}^n \overline{\alpha}_i \partial_{\bar{z}_i}\chi_w, h\Big) = (\beta\chi_w, h) = \beta\overline{h(w)}. \tag{5}$$

Substituting consecutively the constant function 1 and the functions $z \mapsto z_i$ $(1 \leqslant i \leqslant n)$ for h, we see that $\beta = \alpha_1 = \dots = \alpha_n = 0$. Now invoking Theorem 1 establishes the result.

When combined with the result of [1], Theorem 1 implies the following

Theorem 2. *Let Ω be an open subset of $\mathbb{C}$ such that $L^2H(\Omega) \neq \{0\}$. Then, for each $w \in \Omega$, $g_\Omega(w)$ is well defined and is positive definite.*

We conclude with a differential-topological characterization of the positive definiteness of the Bergman tensor.

Theorem 3. *Let Ω be an open subset of $\mathbb{C}^n$ $(n \in \mathbb{N})$ such that $L^2H(\Omega) \neq \{0\}$. Suppose that $K_\Omega(w,w) > 0$ for $w \in \Omega$. Then, if $g_\Omega(w)$ is positive definite, then for every $h \in L^2H(\Omega)$ with $h(w) \neq 0$ the mapping $z \mapsto \chi_z/\overline{h(z)}$ is an immersion in an open neighbourhood of w. Conversely, if for some $h \in L^2H(\Omega)$ with $h(w) \neq 0$ the* *mapping* $z \mapsto \chi_z/\overline{h(z)}$ *is an immersion in an open neighbourhood of* w*, then* $g_\Omega(w)$ *is positive definite.*

Proof. Suppose that $g_\Omega(w)$ is positive definite. Let $h \in L^2H(\Omega)$ be such that $h(w) \neq 0$. Then the mapping $z \mapsto \chi_z/\overline{h(z)}$ is antiholomorphic at w, and for any z in an open neighbourhood of w

$$d(\chi_z/\overline{h(z)}) = (\overline{h(z)})^{-2} \sum_{i=1}^{n} \Big(\overline{h(z)}\partial_{\bar z_i}\chi_z - \overline{\frac{\partial h}{\partial z_i}}(z)\chi_z\Big) d\bar z_i. \tag{6}$$

If

$$\sum_{i=1}^{n} \Big(\overline{h(w)}\partial_{\bar z_i}\chi_w - \overline{\frac{\partial h}{\partial z_i}}(w)\chi_w\Big)\overline{\alpha}_i = 0 \tag{7}$$

for some $\alpha_1 \dots, \alpha_n \in \mathbb{C}$, then, by Theorem 1,

$$\overline{h(w)}\overline{\alpha}_1 = \dots \overline{h(w)}\overline{\alpha}_n = \sum_{i=1}^{n} \overline{\frac{\partial h}{\partial z_i}}\overline{\alpha}_i = 0,$$

and so $\alpha_1 = \dots = \alpha_n = 0$. Thus the differential of the mapping $z \mapsto \chi_z/\overline{h(z)}$ at w is injective and hence the mapping itself is an immersion in an open neighbourhood of w.

Suppose now that $h \in L^2H(\Omega)$ is such that $h(w) \neq 0$ and that the mapping $z \mapsto \chi_z/\overline{h(z)}$ is an immersion in an open neighbourhood of w. If (4) holds for some $\alpha_1, \dots, \alpha_n, \beta \in \mathbb{C}$, then, in view of (5), (7) also holds. Taking into account (6) and the fact that the differential of the mapping $z \mapsto \chi_z/\overline{h(z)}$ at w is injective, we see that $\alpha_1 = \dots = \alpha_n = 0$. Now the positive definiteness of $g_\Omega(w)$ follows upon applying Theorem 1.

□

References

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