

Reflexivity and James' theorem via vector-valued Banach limits

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Abstract. We present a new, concise proof of the result that a Banach space X is reflexive if it admits a Banach limit on bounded X -valued sequences such that the limit vector of any input sequence lies in the closed linear span of that sequence. Building on this result, and aiming to demonstrate the utility of vector-valued Banach limits with the linear span property just described, we provide a new proof of James' theorem characterising reflexive Banach spaces with a Schauder basis. This proof consists of two parts, both employing vector-valued Banach limits, with one part specifically relying on Banach limits that possess the linear span property.

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1. Introduction

Let X be a vector space over the field of real or complex numbers. Let $\mathbb{N}$ denote the set of natural numbers, and let $X^{\mathbb{N}}$ denote the space of all sequences in X . We typically write a sequence $(x_n)_{n \geq 1} \in X^{\mathbb{N}}$ as (x_n) , with $\bar{x}$ as a shorthand, though we may revert to $(x_n)_{n \geq 1}$ for clarity or emphasis.

We denote by $\mathbb{1}$ the constant sequence $(1, 1, \dots)$. For $x \in X$, we denote by $x\mathbb{1}$ the constant sequence $(x, x, \dots)$.

The backward shift operator, or left shift operator, $\tau: X^{\mathbb{N}} \rightarrow X^{\mathbb{N}}$ is defined by

$$\bar{x} = (x_1, x_2, \dots) \mapsto \tau\bar{x} = (x_2, x_3, \dots).$$

Now suppose that X is a normed space. For a subset $A \subset X$, we denote by $\overline{\text{co}} A$ the closed convex hull of A , and by $\overline{\text{lin}} A$ the closed linear span of A .

Let ℓ_X^∞ denote the space of all bounded X -valued sequences, so that

$$\ell_X^\infty = \left\{ (x_n) \in X^{\mathbb{N}} : \sup_{n \in \mathbb{N}} \|x_n\| < \infty \right\},$$

equipped with the uniform norm $\|\cdot\|_\infty$ defined by

$$\|(x_n)\|_\infty = \sup_{n \in \mathbb{N}} \|x_n\| \quad ((x_n) \in \ell_X^\infty).$$

We write ℓ^∞ for ℓ_X^∞ when X is the scalar field.

We recall that a *Banach limit* is a bounded linear functional m on ℓ^∞ that satisfies the following conditions:

- (i) $m(\mathbf{1}) = 1 = \|m\|$;
- (ii) $m(\tau\bar{\alpha}) = m(\bar{\alpha})$ for all $\bar{\alpha} \in \ell^\infty$.

It is a foundational result, first announced by Mazur [22] and later established by Banach [3], that such a functional exists.

In what follows, we shall consider vector-valued analogues of Banach limits, defined as follows:

Definition 1.1. Let X be a Banach space. A bounded linear mapping $M: \ell_X^\infty \rightarrow X$ is a *X -valued Banach limit* if:

- (i) $M(x\mathbf{1}) = x$ for all $x \in X$;
- (ii) $M(\tau\bar{x}) = M(\bar{x})$ for all $\bar{x} \in \ell_X^\infty$.

This definition has been adopted by several authors (e.g., [6], [15], [16], [21], [26]). However, alternative definitions—similar but distinct—have also been proposed in the literature (e.g., [2], [7], [13], [14], [24], [25], [28]).

Of particular interest for us throughout will be vector-valued Banach limits with additional geometric properties, as described in the following definition:

Definition 1.2. Let X be a Banach space, and let M be a X -valued Banach limit on ℓ_X^∞ . We say that:

- (a) M has the *convex range property*, or is *convex-range*, if

$$M(\bar{x}) \in \overline{\text{co}}\{x_n : n \in \mathbb{N}\} \quad \text{for all } \bar{x} = (x_n) \in \ell_X^\infty;$$

- (b) M has the *linear span property*, or is *linearly generated*, if

$$M(\bar{x}) \in \overline{\text{lin}}\{x_n : n \in \mathbb{N}\} \quad \text{for all } \bar{x} = (x_n) \in \ell_X^\infty.$$

It is well known that every scalar-valued Banach limit has the convex range property (see, e.g., [17, pp. 1–2]). One consequence of this fact is that every scalar-valued Banach limit is a positive linear functional; that is, if m is a Banach limit and $\bar{x} = (x_n) \in \ell^\infty$ is such that $x_n \geq 0$ for each $n \in \mathbb{N}$, then $m(\bar{x}) \geq 0$. Another consequence is that every scalar-valued Banach limit is reality-preserving, meaning that it takes real values on real-valued sequences.

It is also a standard result that if X is a reflexive Banach space, then there exists a convex-range X -valued Banach limit on ℓ_X^∞ . In the proof of Theorem 3.1 below, we shall reproduce the relevant construction as part of the argument presented there. This result is a special case of more general

theorems formulated in the setting of X -valued (left, right) invariant means on bounded X -valued functions on (left, right) amenable semigroups, with X being a reflexive Banach space (see, e.g., [9], [18], [19], [12]).

Conversely, the existence of a convex-range X -valued Banach limit on ℓ_X^∞ for a Banach space X implies that X is reflexive (cf. [4], [5], [27]). This result was further extended in [6], where it was shown that a Banach space X is reflexive if there exists a linearly generated X -valued Banach limit on ℓ_X^∞ .

In this note, we present a new, concise proof of the result from [6]. We also apply this result to provide a new proof of James' theorem, which characterises reflexive Banach spaces with a Schauder basis. In reproving James' theorem, we demonstrate the effectiveness of linearly generated Banach limits. Specifically, the proof of James' theorem is divided into two parts, both of which employ vector-valued Banach limits, with one part relying on Banach limits that possess the linear span property.

2. A new proof of the Banach limit reflexivity criterion

In [6], the following theorem is established:

Theorem 2.1. *Let X be a Banach space. If there exists a linearly generated X -valued Banach limit on ℓ_X^∞ , then X is reflexive.*

In this section, we present an alternative proof of Theorem 2.1. The original proof relies on Pelczyński's version of the Eberlein–Šmulian theorem, and specifically on a characterisation of weak compactness in Banach spaces via basic sequences (see [23] and [8, p. 41] for a textbook presentation). In contrast, our proof uses the Eberlein–Šmulian theorem in its standard form (see, e.g., [8, p. 18], [10, p. 430], [29], and [30, p. 50] for various proofs). The standard form, just like Pelczyński's version, offers several equivalent characterisations of weak compactness, and we exploit the one involving weakly convergent subsequences.

We begin by introducing some notation and establishing an auxiliary lemma.

Let X be a normed space. We denote by B_X the closed unit ball of X , so that $B_X = \{x \in X : \|x\| \leq 1\}$. The dual space of X is denoted by X^* , and the second dual space by X^{**} . The action of $x^* \in X^*$ on an element $x \in X$ is written as $\langle x, x^* \rangle$. The identity operator on X is denoted by I_X .

Lemma 2.2. *Let X be a Banach space, and let M be a linearly generated X -valued Banach limit on ℓ_X^∞ . Suppose that $\bar{x} = (x_n) \in \ell_X^\infty$ and $x^* \in X^*$ are such that the sequence $(\langle x_n, x^* \rangle)$ converges. Then*

$$\lim_{n \rightarrow \infty} \langle x_n, x^* \rangle = \langle M(\bar{x}), x^* \rangle.$$

Proof. If x^* is the zero functional, the result is trivial. Suppose that $x^* \neq 0$. Then there exists $x \in X$ such that $\langle x, x^* \rangle = 1$. For each $n \in \mathbb{N}$, define

$$y_n = x_n - \langle x_n, x^* \rangle x.$$

The sequence $\bar{y} = (y_n)$ belongs to ℓ_X^∞ , and $\langle y_n, x^* \rangle = \langle x_n, x^* \rangle - \langle x_n, x^* \rangle \langle x, x^* \rangle = 0$ for all $n \in \mathbb{N}$. Since, by virtue of M being linearly generated, $M(\bar{y}) \in \overline{\text{lin}}\{y_n : n \in \mathbb{N}\}$, it follows that

$$\langle M(\bar{y}), x^* \rangle = 0. \quad (2.1)$$

Let $g = \lim_{n \rightarrow \infty} \langle x_n, x^* \rangle$. Since the sequence $(\langle x_n, x^* \rangle x)$ converges to gx , and X -valued Banach limits agree with the usual limit on convergent sequences, we have

$$M((\langle x_n, x^* \rangle x)) = gx.$$

Thus

$$M(\bar{y}) = M(\bar{x}) - M((\langle x_n, x^* \rangle x)) = M(\bar{x}) - gx.$$

Applying x^* to both sides, we obtain

$$\langle M(\bar{y}), x^* \rangle = \langle M(\bar{x}), x^* \rangle - g.$$

Combining this with (2.1), we conclude that $\langle M(\bar{x}), x^* \rangle = g$, as required. $\square$

Proof of Theorem 2.1. Suppose that M is a linearly generated X -valued Banach limit on ℓ_X^∞ . To prove that X is reflexive, it suffices to show, by a standard consequence of the Eberlein–Šmulian theorem (see, e.g., [1, Corollary 1.6.4]), that every bounded sequence in X admits a subsequence that converges weakly to an element of X .

Let $(x_n) \in \ell_X^\infty$, and set $Y = \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$. It is immediate that if $(y_n) \in \ell_Y^\infty$, then $\overline{\text{lin}}\{y_n : n \in \mathbb{N}\} \subset Y$. Since M is linearly generated, it follows that M maps ℓ_Y^∞ into Y . Moreover, Y is separable, so Y^* is separable in the $\sigma(Y^*, Y)$ -topology (see, e.g., [11, Corollary 3.104]). Let $\{y_k^* : k \in \mathbb{N}\}$ be a $\sigma(Y^*, Y)$ -dense subset of Y^* . Since (x_n) is bounded, the scalar sequence $(\langle x_n, y_k^* \rangle)_{n \geq 1}$ is bounded for each $k \in \mathbb{N}$. By a standard diagonal argument, there exists a subsequence $(x_{n_m})_{m \geq 1}$ such that $(\langle x_{n_m}, y_k^* \rangle)_{m \geq 1}$ converges for every $k \in \mathbb{N}$.

By Lemma 2.2, we have

$$\lim_{m \rightarrow \infty} \langle x_{n_m}, y_k^* \rangle = \langle M((x_{n_m})_{m \geq 1}), y_k^* \rangle \quad (2.2)$$

for all $k \in \mathbb{N}$. We shall show next that, for every $y^* \in Y^*$,

$$\lim_{m \rightarrow \infty} \langle x_{n_m}, y^* \rangle = \langle M((x_{n_m})_{m \geq 1}), y^* \rangle. \quad (2.3)$$

Since, for any $x^* \in X^*$, the restriction $x^*|_Y$ of x^* to Y belongs to Y^* , this will imply that, for every $x^* \in X^*$,

$$\begin{aligned} \lim_{m \rightarrow \infty} \langle x_{n_m}, x^* \rangle &= \lim_{m \rightarrow \infty} \langle x_{n_m}, x^*|_Y \rangle \\ &= \langle M((x_{n_m})_{m \geq 1}), x^*|_Y \rangle = \langle M((x_{n_m})_{m \geq 1}), x^* \rangle, \end{aligned}$$

which will complete the proof.

Fix $y^* \in Y^*$. Since the sequence $(\langle x_{n_m}, y^* \rangle)_{m \geq 1}$ is bounded, to establish (2.3), it suffices to show that every convergent subsequence

$(\langle x_{n_{m_p}}, y^* \rangle)_{p \geq 1}$ has the same limit $\langle M((x_{n_m})_{m \geq 1}), y^* \rangle$. Suppose that a subsequence $(\langle x_{n_{m_p}}, y^* \rangle)_{p \geq 1}$ converges. By Lemma 2.2, we have

$$\lim_{p \rightarrow \infty} \langle x_{n_{m_p}}, y^* \rangle = \langle M((x_{n_{m_p}})_{p \geq 1}), y^* \rangle. \quad (2.4)$$

For each $k \in \mathbb{N}$, since $(\langle x_{n_m}, y_k^* \rangle)_{m \geq 1}$ converges, the subsequence $(\langle x_{n_{m_p}}, y_k^* \rangle)_{p \geq 1}$ converges to the same limit:

$$\lim_{p \rightarrow \infty} \langle x_{n_{m_p}}, y_k^* \rangle = \lim_{m \rightarrow \infty} \langle x_{n_m}, y_k^* \rangle. \quad (2.5)$$

Applying Lemma 2.2 again, we obtain

$$\lim_{p \rightarrow \infty} \langle x_{n_{m_p}}, y_k^* \rangle = \langle M((x_{n_{m_p}})_{p \geq 1}), y_k^* \rangle$$

for all $k \in \mathbb{N}$. This, together with (2.2) and (2.5), yields

$$\langle M((x_{n_{m_p}})_{p \geq 1}), y_k^* \rangle = \langle M((x_{n_m})_{m \geq 1}), y_k^* \rangle$$

for all $k \in \mathbb{N}$. Since $\{y_k^* : k \in \mathbb{N}\}$ is $\sigma(Y^*, Y)$ -dense in Y^* , we deduce that

$$\langle M((x_{n_{m_p}})_{p \geq 1}), y^* \rangle = \langle M((x_{n_m})_{m \geq 1}), y^* \rangle.$$

Combining this with (2.4), we finally conclude that

$$\lim_{p \rightarrow \infty} \langle x_{n_{m_p}}, y^* \rangle = \langle M((x_{n_m})_{m \geq 1}), y^* \rangle,$$

as required. $\square$

3. A new proof of James' theorem

In this section, we apply Theorem 2.1 to provide a new proof of James' theorem characterising reflexive Banach spaces with a Schauder basis. The goal is to demonstrate the utility of linearly generated Banach limits. The proof is divided into two parts, both relying on vector-valued Banach limits. One part specifically employs the linearly generated variant, thereby highlighting its importance in this context.

Recall that a sequence (e_n) in a Banach space X is a *Schauder basis*, or simply a *basis*, for X if, for every $x \in X$, there exists a unique sequence of scalars (a_n) such that $\sum_{i=1}^{\infty} a_i e_i$ converges in norm to x . For each $n \in \mathbb{N}$, the mapping $x \mapsto a_n$ then defines a continuous linear functional e_n^* on X , and the identity $\langle e_m, e_n^* \rangle = \delta_{m,n}$, where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise, holds for all $m, n \in \mathbb{N}$. Moreover, every $x \in X$ admits the representation

$$x = \sum_{i=1}^{\infty} \langle x, e_i^* \rangle e_i,$$

where the series converges in norm. For each $n \in \mathbb{N}$, the map

$$P_n : X \rightarrow X, \quad \sum_{i=1}^{\infty} \langle x, e_i^* \rangle e_i \mapsto \sum_{i=1}^n \langle x, e_i^* \rangle e_i,$$

is a projection onto the linear span of $\{e_i : 1 \leq i \leq n\}$, called the *n-th natural projection*. The natural projections are bounded linear operators and,

by the Banach–Steinhaus theorem, satisfy $\sup_{n \in \mathbb{N}} \|P_n\| < \infty$. The quantity $\sup_{n \in \mathbb{N}} \|P_n\|$ is called the *basis constant* for (e_n) .

A basis (e_n) in a Banach space X is called *shrinking* if, for every $x^* \in X^*$,

$$\lim_{n \rightarrow \infty} \left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq n\}} \right\| = 0,$$

where $\left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq n\}} \right\|$ is the norm of the restriction of x^* to $\overline{\text{lin}}\{e_i : i \geq n\}$, that is,

$$\left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq n\}} \right\| = \sup \{ |\langle y, x^* \rangle| : y \in \overline{\text{lin}}\{e_i : i \geq n\}, \|y\| \leq 1 \}.$$

Note, incidentally, that the sequence $\left(\left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq n\}} \right\| \right)_{n \geq 1}$ is non-increasing for each $x^* \in X^*$. A basis (e_n) is said to be *boundedly complete* if, for every sequence of scalars (a_n) such that $\sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^n a_i e_i \right\| < \infty$, the series $\sum_{i=1}^{\infty} a_i e_i$ converges in X .

Theorem 3.1 (James [20]). *Let X be a Banach space with a basis (e_n) . Then X is reflexive if and only if (e_n) is both shrinking and boundedly complete.*

Proof. Necessity. Suppose that X is reflexive. We shall show that (e_n) is both shrinking and boundedly complete. Let m be a Banach limit on ℓ^∞ . For any $\bar{x} = (x_n) \in \ell^\infty(X)$ and $x^* \in X^*$, we have

$$|m(\langle x_n, x^* \rangle)| \leq \|(\langle x_n, x^* \rangle)\|_\infty \leq \|\bar{x}\|_\infty \|x^*\|.$$

Thus the functional $x^* \mapsto m(\langle x_n, x^* \rangle)$ belongs to X^{**} . Since X is reflexive, there exists a unique $M(\bar{x}) \in X$ such that

$$m(\langle x_n, x^* \rangle) = \langle M(\bar{x}), x^* \rangle \quad (x^* \in X^*). \quad (3.1)$$

It is readily verified that the mapping

$$M : \ell^\infty(X) \rightarrow X, \quad \bar{x} \mapsto M(\bar{x}),$$

is an X -valued Banach limit. Although not essential for what follows, we note that M possesses the convex range property. To see this, assume that there exists $\bar{x} = (x_n) \in \ell^\infty(X)$ such that $M(\bar{x}) \notin \overline{\text{co}}\{x_n : n \in \mathbb{N}\}$. Then, by the Hahn–Banach separation theorem (see, e.g., [11, Theorem 3.32]), there exists $x^* \in X^*$ such that

$$\sup_{n \in \mathbb{N}} \text{Re} \langle x_n, x^* \rangle < \text{Re} \langle M(\bar{x}), x^* \rangle.$$

However, by the reality-preserving and positivity properties of m ,

$$\text{Re} \langle M(\bar{x}), x^* \rangle = \text{Re} m(\langle x_n, x^* \rangle) = m(\text{Re} \langle x_n, x^* \rangle) \leq \sup_{n \in \mathbb{N}} \text{Re} \langle x_n, x^* \rangle,$$

a contradiction. Hence M has the convex range property.

To prove that (e_n) is shrinking, assume towards a contradiction that this is not the case. Then there exists $x^* \in X^*$ such that

$$\lim_{n \rightarrow \infty} \left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq n\}} \right\| = \delta > 0.$$

Since the sequence $\left(\left\|x^*|_{\overline{\text{lin}}\{e_i : i \geq n\}}\right\|\right)_{n \geq 1}$ is non-increasing, we have

$$\left\|x^*|_{\overline{\text{lin}}\{e_i : i \geq n\}}\right\| \geq \delta$$

for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, select $x_n \in \overline{\text{lin}}\{e_i : i \geq n\}$ with $\|x_n\| \leq 1$ and $|\langle x_n, x^* \rangle| \geq \delta/2$. If necessary, replace x_n with $(\langle x_n, x^* \rangle / |\langle x_n, x^* \rangle|)x_n$ to ensure that $\langle x_n, x^* \rangle \geq \delta/2$. Consequently, $m(\langle x_n, x^* \rangle) \geq \delta/2$, which, by (3.1), can be restated as

$$\langle M(\bar{x}), x^* \rangle \geq \delta/2. \quad (3.2)$$

However, since $\langle x_n, e_k^* \rangle = 0$ for $n > k$, we have

$$m(\langle x_n, e_k^* \rangle)_{n \geq 1} = 0$$

for each $k \in \mathbb{N}$. This, again by (3.1), amounts to

$$\langle M(\bar{x}), e_k^* \rangle = 0$$

for all $k \in \mathbb{N}$, so $M(\bar{x}) = 0$, contradicting (3.2). Hence (e_n) is shrinking.

To prove that (e_n) is boundedly complete, consider a sequence of scalars (a_n) such that $(\sum_{i=1}^n a_i e_i)_{n \geq 1}$ is bounded. For each $k \in \mathbb{N}$ and each $n \geq k$, we have

$$\left\langle \sum_{i=1}^n a_i e_i, e_k^* \right\rangle = a_k,$$

so that

$$m \left(\left(\left\langle \sum_{i=1}^n a_i e_i, e_k^* \right\rangle \right)_{n \geq 1} \right) = a_k$$

for each $k \in \mathbb{N}$. By (3.1), this yields

$$\left\langle M \left(\left(\sum_{i=1}^n a_i e_i \right)_{n \geq 1} \right), e_k^* \right\rangle = a_k$$

for all $k \in \mathbb{N}$. The latter means that the series $\sum_{i=1}^{\infty} a_i e_i$ converges to $M \left(\left(\sum_{i=1}^n a_i e_i \right)_{n \geq 1} \right)$, proving that (e_n) is boundedly complete.

Sufficiency. Suppose (e_n) is both shrinking and boundedly complete. We shall show that X is reflexive by constructing a linearly generated Banach limit on ℓ_X^∞ , which, by Theorem 2.1, ensures reflexivity.

Let m be a Banach limit on ℓ^∞ . For each $k \in \mathbb{N}$, define $M_k : \ell_X^\infty \rightarrow X$ by

$$M_k(\bar{x}) := \sum_{i=1}^k m(\langle x_n, e_i^* \rangle)_{n \geq 1} e_i \quad (\bar{x} = (x_n) \in \ell_X^\infty).$$

Then, for any $\bar{x} = (x_n) \in \ell_X^\infty$ and $x^* \in X^*$,

$$\langle M_k(\bar{x}), x^* \rangle = \sum_{i=1}^k m(\langle x_n, e_i^* \rangle)_{n \geq 1} \langle e_i, x^* \rangle = m(\langle P_k x_n, x^* \rangle)_{n \geq 1},$$

where P_k denotes the k -th natural projection associated with (e_n) . It follows that

$$\begin{aligned} \|M_k(\bar{x})\| &= \sup_{x^* \in B_{X^*}} |\langle M_k(\bar{x}), x^* \rangle| \leq \sup_{x^* \in B_{X^*}} m((|\langle P_k x_n, x^* \rangle|)_{n \geq 1}) \\ &\leq \sup_{n \in \mathbb{N}} \|P_k x_n\| \leq \sup_{k \in \mathbb{N}} \|P_k\| \cdot \|\bar{x}\|_\infty, \end{aligned}$$

so the sequence $(M_k(\bar{x}))_{k \geq 1}$ is bounded. Since (e_n) is boundedly complete, the limit

$$M(\bar{x}) := \lim_{k \rightarrow \infty} M_k(\bar{x}) = \sum_{i=1}^{\infty} m((\langle x_n, e_i^* \rangle)_{n \geq 1}) e_i \quad (3.3)$$

exists. A routine verification shows that the map

$$M: \ell_X^\infty \rightarrow X, \quad \bar{x} \mapsto M(\bar{x}),$$

is an X -valued Banach limit.

We now prove that M is linearly generated. Let $\bar{x} = (x_n) \in \ell_X^\infty$. To establish that $M(\bar{x}) \in \overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$, it suffices, by the Hahn–Banach theorem, to show that if $x^* \in X^*$ satisfies $\langle x_n, x^* \rangle = 0$ for all $n \in \mathbb{N}$, then $\langle M(\bar{x}), x^* \rangle = 0$.

So suppose $x^* \in X^*$ is such that $\langle x_n, x^* \rangle = 0$ for all $n \in \mathbb{N}$. Fix $\epsilon > 0$, and choose $k_0 \in \mathbb{N}$ such that $\left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq k_0 + 1\}} \right\| \leq \epsilon$. For each $k \in \mathbb{N}$ with $k \geq k_0$ and each $n \in \mathbb{N}$, we have

$$\begin{aligned} |\langle (I_X - P_k)x_n, x^* \rangle| &\leq \|(I_X - P_k)x_n\| \cdot \left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq k+1\}} \right\| \\ &\leq (1 + \|P_k\|)\|x_n\| \cdot \left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq k+1\}} \right\| \\ &\leq (1 + C)\|\bar{x}\|_\infty \cdot \left\| x^*|_{\overline{\text{lin}}\{e_i : i \geq k_0 + 1\}} \right\| \\ &\leq \epsilon(1 + C)\|\bar{x}\|_\infty, \end{aligned}$$

where $C = \sup_{n \in \mathbb{N}} \|P_n\|$ is the basis constant of (e_n) . Since $\langle x_n, x^* \rangle = 0$, we also have

$$|\langle (I_X - P_k)x_n, x^* \rangle| = |\langle P_k x_n, x^* \rangle|,$$

and hence

$$|\langle P_k x_n, x^* \rangle| \leq \epsilon(1 + C)\|\bar{x}\|_\infty.$$

Thus

$$|\langle M_k(\bar{x}), x^* \rangle| \leq m((|\langle P_k x_n, x^* \rangle|)_{n \geq 1}) \leq \epsilon(1 + C)\|\bar{x}\|_\infty.$$

Letting $k \rightarrow \infty$ and invoking (3.3), we obtain

$$|\langle M(\bar{x}), x^* \rangle| \leq \epsilon(1 + C)\|\bar{x}\|_\infty.$$

Since $\epsilon > 0$ was arbitrary, it follows that $\langle M(\bar{x}), x^* \rangle = 0$, as required. $\square$

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