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Almost Periodic Schrödinger Operators in $L^2(b\mathbb{R})$ Whose Point Spectrum Is Not All of the Spectrum

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We exhibit almost periodic potentials such that the corresponding Schrödinger operators in the space of all square Haar-integrable functions on the Bohr compactification of $\mathbb{R}$ have a point in the spectrum which is not an eigenvalue.

1. INTRODUCTION

In studying Schrödinger operators with potentials that are real (uniformly) almost periodic function, apart from the usual L^2 analysis, one performs the so-called quasi-momentum analysis of these operators, examining their action in the nonseparable Hilbert space of all square Haar-integrable functions on the Bohr compactification $b\mathbb{R}^n$ of $\mathbb{R}^n$. The general belief is that the quasi-momentum approach is more adequate for handling pseudodifferential operators with (spatially) almost periodic symbols, and enables one to yield new and reestablish more directly known results of the usual L^2 theory (cf. [1, 2, 7, 9]). As far as Schrödinger operators are concerned, an example of a result of the first kind is the spectral mixing theorem (cf. [3, 7]); an example of a result of the second kind being one to the effect that the Bloch waves (generalized eigenfunctions) for a Schrödinger operator in $L^2(\mathbb{R}^n)$ with a periodic potential can be recovered from a complete system of eigenelements of the corresponding operator in $L^2(b\mathbb{R}^n)$ (cf. [1, 7]).

Up to now the structure of almost periodic Schrödinger operators in $L^2(b\mathbb{R}^n)$ has not been fully explored. Burnat [1] showed that in the case of a one-dimensional periodic potential, the point spectrum (the totality of all eigenvalues) of the corresponding Schrödinger operator in $L^2(b\mathbb{R}^n)$ is all of the spectrum (this is a stronger property than that for the operator to have pure point spectrum, which merely means that the eigenelements span the

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space). Below we exhibit one-dimensional almost periodic potentials such that the corresponding Schrödinger operators in $L^2(b\mathbb{R})$ have a point in the spectrum outside the point spectrum. It will remain an open question whether these operators actually have a continuous component in the spectrum.

2. Prerequisites

Let $\mathbb{P}$ be the probabilistic Haar measure on $b\mathbb{R}$, and $L^p(b\mathbb{R})$ $(1 \leqslant p \leqslant +\infty)$ be the Lebesgue spaces based on $\mathbb{P}$. Let $\hat{}$ be the Fourier transform that unitarily maps $L^2(b\mathbb{R})$ onto $l^2(\mathbb{R})$.

There is a kind of differential operator on $b\mathbb{R}$, D, whose domain consists of all H's in $L^2(b\mathbb{R})$ such that $\sum_{\mu \in \mathbb{R}} \mu^2 \, |\hat{H}(\mu)|^2 < +\infty$, and whose action on any element H in the domain is uniquely determined by

$$DH^\wedge(\mu) = i\mu\hat{H}(\mu) \qquad (\mu \in \mathbb{R}).$$

As an operator in $L^2(b\mathbb{R})$, D is unbounded and skew-adjoint. By analogy with $i^{-1}(d/dx)$, $i^{-1}D$ is called the quasi-momentum operator.

Let $AP_{\mathbb{R}}(\mathbb{R})$ be the space of all real (uniformly) almost periodic functions on $\mathbb{R}$.

With a function q in $AP_{\mathbb{R}}(\mathbb{R})$, one associates the one-dimensional Schrödinger operator

$$l = -\frac{d^2}{dx^2} + q$$

having the Sobolev space $H^2(\mathbb{R})$ as domain, which is unbounded and self-adjoint as an operator in $L^2(\mathbb{R})$. Let α be the canonical homomorphism from $\mathbb{R}$ into $b\mathbb{R}$, and Q be the unique real continuous function on $b\mathbb{R}$ such that $Q(\alpha(x)) = q(x))$ for all $x \in \mathbb{R}$. Q gives rise to the analogue of l on $b\mathbb{R}$, the operator

$$L = -D^2 + Q$$

whose domain is that of D^2, i.e., the subspace of all H's in $L^2(b\mathbb{R})$ satisfying $\sum_{\mu \in \mathbb{R}} \mu^4 \, |\hat{H}(\mu)|^2 < +\infty$. As an operator in $L^2(b\mathbb{R})$, L is unbounded and self-adjoint. The spectral analysis of L is precisely what one means by the quasi-momentum analysis of l (the correspondence between l and L is obviously one-to-one).

The subsequent quasi-momentum analysis will be based upon a result on cocycles from the theory of invariant subspaces on compact Abelian groups with ordered duals (cf. [4, 5]).

Henceforth, given a function h on an Abelian group and an element s of the group, we shall denote by h_s the translate of h by s; if h is a function on $b\mathbb{R}$ and $s \in \mathbb{R}$, we shall write h_s instead of $h_{\alpha(s)}$.

If f is a real almost periodic function on $\mathbb{R}$ and F its continuous extension on $b\mathbb{R}$, then the function

$$Y_t(\omega) = \exp\left(i \int_0^t F_u(\omega)\, du\right) \qquad (t \in \mathbb{R}, \omega \in b\mathbb{R})$$

is an example of what one calls a cocycle (here, of course, Y_t is not the translate by $\alpha(t)$ of any Y!). Throughout it will be convenient to look upon this functions as a (wide sense stationary) process $\{Y_t\}$ carried by $(b\mathbb{R}, \mathbb{P})$. Using the Trotter product formula, one shows that the strongly continuous unitary group $\{\mathscr{U}_t\}$ in $L^2(b\mathbb{R})$ generated by the operator $D + iF$ (whose domain is that of D) can be represented in the form

$$\mathscr{U}_t H = Y_t \cdot H_t \qquad (t \in \mathbb{R}, H \in L^2(b\mathbb{R})). \tag{1}$$

One proves (cf. [4]), moreover, that if f is of the form

$$f(x) = \sum_{k=1}^{\infty} \alpha_k\, p(\alpha_k x) \qquad (x \in \mathbb{R}),$$

where (α_k) is a sequence of rationally independent real numbers such that $\sum_{k=1}^{\infty} |\alpha_k| < +\infty$, and p is non-zero real continuous periodic function on $\mathbb{R}$ with mean value zero, then the self-adjoint operator $i^{-1}D + F$ has a purely continuous spectrum. Of course, adding a real constant to f does not change the type of spectrum of the resulting first-order differential operator on $b\mathbb{R}$. Taking p to be twice continuously differentiable gives f with almost periodic first and second derivatives. Thus the class of functions f in $AP_{\mathbb{R}}(\mathbb{R})$ such that $f', f'' \in AP_{\mathbb{R}}(\mathbb{R})$, $f \geqslant c > 0$ for some $c > 0$, and the operator $i^{-1}D + F$ has purely continuous spectrum, is nonvoid. Henceforth, any element of this class will be called an admissible almost periodic function on $\mathbb{R}$.

3. The Result

We shall prove the following.

Theorem. *If f is an admissible almost periodic function on $\mathbb{R}$ and q is the real almost periodic function defined* as

$$q = \tfrac{3}{4} f^{-2} f'^2 - \tfrac{1}{2} f^{-1} f'' - f^2,$$

then 0 *is in the spectrum of*

$$L = -D^2 + Q,$$

where Q *is the continuous extension of* q *on* $b\mathbb{R}$, *and the kernel of* L *is* $\{0\}$.

Proof. The function

$$x \to f^{-1/2}(x) \exp\left(i \int_0^x f(u)\, du\right) \qquad (x \in \mathbb{R})$$

is a generalized eigenfunction of the operator

$$l = -\frac{d^2}{dx^2} + q,$$

corresponding to the eigenvalue 0; thus 0 is in the spectrum of l. By a theorem of Šubin [8], in the version given by Herczyński [6] that covers the case of almost periodic potentials which are not necessarily C^∞, the spectra of l and L coincide. Consequently, 0 belongs to the spectrum of L.

Let X be a function in the domain of L such that

$$D^2 X = QX. \tag{2}$$

Note first that the mapping $\mathbb{R} \ni t \to X_t \in L^2(b\mathbb{R})$ is twice differentiable and, moreover,

$$\frac{d^2 X_t}{dt^2} = (D^2 X)_t. \tag{3}$$

Indeed, by Plancherel's theorem, for any $t \in \mathbb{R}$ and any $h \in \mathbb{R} \setminus \{0\}$, we have

$$\left\| \frac{X_{t+h} - X_t}{h} - (DX)_t \right\|_2 = \sum_{\mu \in \mathbb{R}} \left| \left(\frac{e^{i\mu(t+h)} - e^{i\mu t}}{h} - i\mu e^{i\mu t} \right) \hat{X}(\mu) \right|^2$$
$$= \sum_{\mu \in \mathbb{R}} \left| \frac{e^{i\mu h} - 1}{h} - i\mu \right|^2 |\hat{X}(\mu)|^2.$$

Taking into account that for any $\mu \in \mathbb{R}$ and any $h \in \mathbb{R} \setminus \{0\}$

$$\left| \frac{e^{i\mu h} - 1}{h} - i\mu \right| \leqslant 2\, |\mu|,$$

and, of course, $\sum_{\mu \in \mathbb{R}} \mu^2\, |\hat{X}(\mu)|^2 < +\infty$, the equality

$$\frac{dX_t}{dt} = (DX)_t \tag{4}$$

follows upon applying Lebesgue's dominated convergence theorem. Repeating the argument, we obtain (3).

Equation (2) jointly with (3) yields

$$\frac{d^2X_t}{dt^2}=Q_tX_t. \tag{5}$$

Let F, F', F'' be the continuous extension on $b\mathbb{R}$ of f, f', f'', respectively. Of course, $F\geqslant c>0$ for some $c>0$, and $F'=DF$ and $F''=D^2F$. From now onwards, we shall write f'_t, F'_t, etc., instead of $(f')_t$, $(F')_t$, etc. Since for any $h\in\mathbb{R}\setminus\{0\}$

$$\left\|\frac{f_h-f}{h}-f'\right\|_\infty\leqslant\sup_{0\leqslant|\xi|\leqslant|h|}\|f'_\xi-f'\|_\infty,$$

we see that the expression

$$\left\|\frac{F_{t+h}-F_t}{h}-F'_t\right\|_\infty=\left\|\frac{F_h-F}{h}-F'\right\|_\infty=\left\|\frac{f_h-f}{h}-f'\right\|_\infty$$

tends to zero as $h\to 0$. Thus

$$\frac{dF_t}{dt}=F'_t,$$

the differentiation on the left-hand side being taken in the supremum norm. Analogously

$$\frac{d^2F_t}{dt^2}=F''_t.$$

For any $t\in\mathbb{R}$ and any $\omega\in b\mathbb{R}$, set

$$E_t^\pm(\omega)=F_t^{-1/2}(\omega)\exp\left(\pm i\int_0^t F_u(\omega)\,du\right).$$

Using the preceding two equalities, one verifies at once that each of the functions $\mathbb{R}\ni t\to E_t^\pm\in C(b\mathbb{R})$, $C(b\mathbb{R})$ being the space of all continuous functions on $b\mathbb{R}$, is twice differentiable and

$$\frac{dE_t^\pm}{dt}=(\pm iF_t^{1/2}-\tfrac{1}{2}F_t^{-3/2}F'_t)\exp\left(\pm i\int_0^t F_u\,du\right),$$

$$\frac{d^2E_t^\pm}{dt^2}=Q_tE_t^\pm.$$

For any $t \in \mathbb{R}$, put

$$Z_t = AE_t^+ + BE_t^-,$$

where

$$A = \tfrac{1}{2}[F^{1/2}X - i(F^{-1/2}\,DX + \tfrac{1}{2}F^{-3/2}F'X)],$$
$$B = \tfrac{1}{2}[F^{1/2}X + i(F^{-1/2}\,DX + \tfrac{1}{2}F^{-3/2}F'X)].$$

A straightforward calculation shows that $\mathbb{R} \ni t \to Z_t \in L^2(b\mathbb{R})$ is a solution of the following Cauchy problem:

$$\frac{d^2 Z_t}{dt^2} = Q_t Z_t,$$

$$Z_0 = X,$$

$$\left(\frac{dZ_t}{dt}\right)_{t=0} = DX.$$

On the other hand, by (4) and (5), $\mathbb{R} \ni t \to X_t \in L^2(b\mathbb{R})$ is also a solution to this problem. As, in view of the uniform boundedness of $\mathbb{R} \ni t \to Q_t \in C(b\mathbb{R})$, the problem may have only one solution, we infer that the processes $\{X_t\}$ and $\{Z_t\}$ are stochastically equivalent: the identity

$$X_t(\omega) = Z_t(\omega)$$

holds for almost all $\omega \in b\mathbb{R}$; in short $\{X_t\} = \{Z_t\}$.

Let $\{Y_t\}$ be the cocycle defined as

$$\{Y_t\} = \left\{\exp\left(i\int_0^t F_u\,du\right)\right\}.$$

It follows from what has preceded that

$$\{F_t^{1/2} X_t\} = \{AY_t + B\bar{Y}_t\}. \tag{6}$$

For any $\mu \in \mathbb{R}$, let χ_μ denote the continuous character of $b\mathbb{R}$ such that $\chi_\mu(\alpha(x)) = e^{i\mu x}$ for all $x \in \mathbb{R}$. Let $\{\mathscr{U}_t\}$ be the unitary group generated by the operator $D + iF$, and, given G, $H \in L^2(b\mathbb{R})$, let $\eta_{G,H}$ be the corresponding spectral measure associated with the operator $i^{-1}D + F$. By (1), for any μ, $t \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{E}(AY_t(\chi_\mu)_t) &= \mathbb{E}((\mathscr{U}_t \chi_\mu)\,A) = \hat{\eta}_{\chi_\mu,\bar{A}}(-t), \\ \mathbb{E}(B\bar{Y}_t(\chi_\mu)_t) &= \mathbb{E}(B\overline{(\mathscr{U}_t \chi_{-\mu})}) = \mathbb{E}((\mathscr{U}_{-t} B)\,\bar{\chi}_{-\mu}) = \hat{\eta}_{B,\chi_{-\mu}}(t); \end{aligned} \tag{7}$$

here $\hat{}$ stands for the Fourier transform of bounded Borel measures on $\mathbb{R}$, and $\mathbb{E}$ denotes the expectation operator relative to $\mathbb{P}$. Since the operator $i^{-1}D+F$ has purely continuous spectrum, it follows that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\hat{\eta}_{\chi_\mu,A}(-t)\,dt=\eta_{\chi_\mu,A}(\{0\})=0,$$
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\hat{\eta}_{B,\chi_{-\mu}}(t)\,dt=\eta_{B,\chi_{-\mu}}(\{0\})=0. \tag{8}$$

(6), (7) and (8) imply that for any $\mu\in\mathbb{R}$

$$\lim_{T\to\infty}\frac{1}{2T}\,\mathbb{E}((F^{1/2}X\chi_\mu)_t)\,dt=0.$$

The flow on $b\mathbb{R}$: $\omega\to\omega+\alpha(t)$ $(t\in\mathbb{R})$ being ergodic, the mean ergodic theorem assures that the expression on the left-hand side equals $\mathbb{E}(F^{1/2}X\chi_\mu)$. As $\{\chi_\mu:\mu\in\mathbb{R}\}$ is a complete orthonormal set in $L^2(b\mathbb{R})$, we infer that $F^{1/2}X=0$, whence $X=0$.

The proof is complete.

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