

A new constrained parameter estimator: experiments in fundamental matrix computation

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Abstract

In recent work the authors proposed a wide-ranging method for estimating parameters that constrain image feature locations and satisfy a constraint not involving image data. The present work illustrates the use of the method with experiments concerning estimation of the fundamental matrix. Results are given for both synthetic and real images. It is demonstrated that the method gives results commensurate with, or superior to, previous approaches, with the advantage of being fast.

1 Introduction

An important problem in computer vision is estimation of the parameters that describe a relationship between image feature locations. In some cases, the parameters are subject to an *ancillary constraint* not involving feature locations. Basic examples include the stereo and motion problems of estimating coefficients of the *epipolar equation* [5] and the *differential epipolar equation* [1], each involving a separate ancillary *cubic constraint*. The *principal equation* applicable in a variety of situations, including those specified above, takes the form

$$\boldsymbol{\theta}^T \mathbf{u}(\mathbf{x}) = 0. \quad (1)$$

Here $\boldsymbol{\theta} = [\theta_1, \dots, \theta_l]^T$ is a vector representing unknown parameters; $\mathbf{x} = [x_1, \dots, x_k]^T$ is a vector representing an element of the data (for example, the locations of a pair of corresponding points); and $\mathbf{u}(\mathbf{x}) = [u_1(\mathbf{x}), \dots, u_l(\mathbf{x})]^T$ is a vector with the data transformed in a problem-dependent manner such that: (i) each component $u_i(\mathbf{x})$ is a quadratic form in the compound vector $[\mathbf{x}^T, 1]^T$, (ii) one component is equal to 1. A common form of the ancillary constraint is

$$\phi(\boldsymbol{\theta}) = 0, \quad (2)$$

where ϕ is a scalar-valued function *homogeneous* of degree κ , i.e. such that

$$\phi(t\boldsymbol{\theta}) = t^\kappa \phi(\boldsymbol{\theta}) \quad (3)$$

for every non-zero scalar t . The estimation problem associated with (1) and (2) can be stated as follows: Given a collection $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ of *image data* and a meaningful *cost*

function that characterises the extent to which any particular θ fails to satisfy (1) with x replaced by x_i ($i = 1, \dots, n$), find $\theta \neq \mathbf{0}$ satisfying (2) for which the cost function attains its minimum. An application of the *Gaussian model of errors* in data in conjunction with the *principle of maximum likelihood* leads to the cost function

$$J_{\text{AML}}(\theta; x_1, \dots, x_n) = \sum_{i=1}^n \frac{\theta^T u(x_i) u(x_i)^T \theta}{\theta^T \partial_x u(x_i) \Lambda_{x_i} \partial_x u(x_i)^T \theta},$$

where, for any length k vector y , $\partial_x u(y)$ denotes the $l \times k$ matrix of the partial derivatives of the function $x \mapsto u(x)$ evaluated at y , and, for each $i = 1, \dots, n$, Λ_{x_i} is a $k \times k$ symmetric *covariance matrix* describing the uncertainty of the data point x_i (see [3, 9]). If J_{AML} is minimised over those non-zero parameter vectors for which (2) holds, then the vector at which the minimum of J_{AML} is attained, the *constrained* minimiser of J_{AML} , defines the *approximated maximum likelihood estimate* $\hat{\theta}_{\text{AML}}$. The *unconstrained* minimiser of J_{AML} obtained by ignoring the ancillary constraint and searching over all of the parameter space defines the *weak approximated maximum likelihood estimate*, $\hat{\theta}_{\text{AML}}^w$. The function $\theta \mapsto J_{\text{AML}}(\theta; x_1, \dots, x_n)$ is homogeneous of degree zero, so both $\hat{\theta}_{\text{AML}}$ and $\hat{\theta}_{\text{AML}}^w$ are determined only up to scale.

Earlier work of the authors [3] presented a method for finding $\hat{\theta}_{\text{AML}}^w$. Recently, the authors proposed a method for calculating $\hat{\theta}_{\text{AML}}$ [4]. The present paper compares these and other methods in the case of fundamental matrix estimation. In light of the results of experiments conducted, the method of constrained minimisation is found to perform better than other methods in terms of accuracy and speed.

2 Fundamental numerical scheme

The unconstrained minimiser $\hat{\theta}_{\text{AML}}^w$ satisfies the *variational equation* for unconstrained minimisation

$$[\partial_\theta J_{\text{AML}}(\theta; x_1, \dots, x_n)]_{\theta=\hat{\theta}_{\text{AML}}^w} = \mathbf{0}^T \quad (4)$$

with $\partial_\theta J_{\text{AML}}$ the row vector of the partial derivatives of J_{AML} with respect to θ . Direct computation shows that

$$[\partial_\theta J_{\text{AML}}(\theta; x_1, \dots, x_n)]^T = 2X_\theta \theta, \quad (5)$$

where

$$X_\theta = \sum_{i=1}^n \frac{A_i}{\theta^T B_i \theta} - \sum_{i=1}^n \frac{\theta^T A_i \theta}{(\theta^T B_i \theta)^2} B_i, \\ A_i = u(x_i) u(x_i)^T, \quad B_i = \partial_x u(x_i) \Lambda_{x_i} \partial_x u(x_i)^T.$$

Thus (4) can be written as

$$[X_\theta \theta]_{\theta=\hat{\theta}_{\text{AML}}^w} = \mathbf{0}. \quad (6)$$

An algorithm for numerically solving this equation proposed in [3] exploits the fact that a vector θ satisfies (6) if and only if it falls into the null space of the matrix X_θ . Thus if θ_{k-1} is a tentative approximate solution, then an improved solution can be obtained

by picking a vector θ_k from that eigenspace of $X_{\theta_{k-1}}$ which most closely approximates the null space of X_θ ; this eigenspace is, of course, the one corresponding to the eigenvalue closest to zero in absolute value. The *fundamental numerical scheme* (FNS) implementing this idea is presented in Figure 1. The scheme is seeded with the *algebraic least squares* (ALS) estimate, $\hat{\theta}_{\text{ALS}}$, defined as the unconstrained minimiser of the cost function $J_{\text{ALS}}(\theta; \mathbf{x}_1, \dots, \mathbf{x}_n) = \|\theta\|^{-2} \sum_{i=1}^n \theta^T \mathbf{A}_i \theta$. The estimate $\hat{\theta}_{\text{ALS}}$ coincides, up to scale, with an eigenvector of $\sum_{i=1}^n \mathbf{A}_i$ associated with the smallest eigenvalue, and this can be found by performing singular-value decomposition (SVD) of the matrix $[\mathbf{u}(\mathbf{x}_1), \dots, \mathbf{u}(\mathbf{x}_n)]$.

1. Set $\theta_0 = \hat{\theta}_{\text{ALS}}$.
2. Assuming θ_{k-1} is known, compute the matrix $X_{\theta_{k-1}}$.
3. Compute a normalised eigenvector of $X_{\theta_{k-1}}$ corresponding to the eigenvalue closest to zero (in absolute value) and take this eigenvector for θ_k .
4. If θ_k is sufficiently close to θ_{k-1} , then terminate the procedure; otherwise increment k and return to Step 2.

Figure 1: Fundamental numerical scheme.

Different but related schemes for numerically solving equations like (6) were developed by Leedan and Meer [10] and Matei and Meer [11]. Yet another approach is Kanatani's [9, Chap. 9] *renormalisation* scheme, in which an estimate is sought at which $\partial_\theta J_{\text{AML}}$ is approximately zero (see [2] for details).

3 Constrained fundamental numerical scheme

By design, FNS does not accommodate the ancillary constraint. One way of enforcing this constraint is to apply some post-hoc correction procedure (see [9, Chap. 5]). In general, however, the modified estimates produced in this way will not coincide with constrained minimisers of the cost function.

An algorithm for determining exact constrained minimisers was proposed in [4]. It is a variant of FNS, in which X_θ is replaced by a more complicated matrix. The scheme is derived starting from the variational equation for constrained minimisation

$$\begin{aligned} [\partial_\theta J_{\text{AML}}(\theta) + \lambda \partial_\theta \phi(\theta)]_{\theta=\hat{\theta}_{\text{AML}}} &= \mathbf{0}^T, \\ \phi(\hat{\theta}_{\text{AML}}) &= 0, \end{aligned} \tag{7}$$

where λ is a suitable Lagrange multiplier. When properly combined with the identity $\partial_\theta \phi(\theta) \theta = \kappa \phi(\theta)$ obtained by differentiating (3) with respect to t and evaluating at $t = 1$, the system (7) can be rewritten as

$$Z_\theta \theta|_{\theta=\hat{\theta}_{\text{AML}}} = \mathbf{0}, \tag{8}$$

where \mathbf{Z}_θ is an $l \times l$ matrix defined as follows. Let \mathbf{P}_θ be the $l \times l$ matrix given by

$$\mathbf{P}_\theta = \mathbf{I}_l - \|\mathbf{a}_\theta\|^{-2} \mathbf{a}_\theta \mathbf{a}_\theta^T,$$

where \mathbf{I}_l denotes the $l \times l$ identity matrix and $\mathbf{a}_\theta = [\partial_\theta \phi(\theta)]^T / 2$. Denote by \mathbf{H}_θ the Hessian of J_{AML} at θ , given explicitly by

$$\mathbf{H}_\theta = 2(\mathbf{X}_\theta - \mathbf{T}_\theta),$$

where

$$\mathbf{T}_\theta = \sum_{i=1}^n \frac{2}{(\theta^T \mathbf{B}_i \theta)^2} \left(\mathbf{A}_i \theta \theta^T \mathbf{B}_i + \mathbf{B}_i \theta \theta^T \mathbf{A}_i - 2 \frac{\theta^T \mathbf{A}_i \theta}{\theta^T \mathbf{B}_i \theta} \mathbf{B}_i \theta \theta^T \mathbf{B}_i \right).$$

Let Φ_θ be the Hessian of ϕ at θ . For each $i \in \{1, \dots, l\}$, let \mathbf{e}_i be the length l vector whose i th entry is unit and all other entries are zero. Now, let

$$\mathbf{Z}_\theta = \mathbf{A}_\theta + \mathbf{B}_\theta + \mathbf{C}_\theta,$$

where

$$\begin{aligned} \mathbf{A}_\theta &= \mathbf{P}_\theta \mathbf{H}_\theta (2\theta \theta^T - \|\theta\|^2 \mathbf{I}_l), \\ \mathbf{B}_\theta &= \|\theta\|^2 \|\mathbf{a}_\theta\|^{-2} \left[\sum_{i=1}^l (\Phi_\theta \mathbf{e}_i \mathbf{a}_\theta^T + \mathbf{a}_\theta \mathbf{e}_i^T \Phi_\theta) \mathbf{X}_\theta \theta \mathbf{e}_i^T - 2 \|\mathbf{a}_\theta\|^{-2} \mathbf{a}_\theta \mathbf{a}_\theta^T \mathbf{X}_\theta \theta \mathbf{a}_\theta^T \Phi_\theta \right], \\ \mathbf{C}_\theta &= \|\mathbf{a}_\theta\|^{-2} \kappa \left[\frac{\phi(\theta)}{4} \Phi_\theta + \mathbf{a}_\theta \mathbf{a}_\theta^T - \frac{\phi(\theta)}{2} \|\mathbf{a}_\theta\|^{-2} \mathbf{a}_\theta \mathbf{a}_\theta^T \Phi_\theta \right]. \end{aligned}$$

Unlike \mathbf{X}_θ , the matrix \mathbf{Z}_θ is not symmetric. To achieve greater resemblance to (6), it proves useful to consider the following equivalent form of (8)

$$[\mathbf{Z}_\theta^T \mathbf{Z}_\theta \theta]_{\theta=\hat{\theta}_{\text{AML}}} = \mathbf{0} \quad (9)$$

with $\mathbf{Z}_\theta^T \mathbf{Z}_\theta$ a symmetric matrix. Now, an algorithm fully analogous to FNS can be advanced by replacing $\mathbf{X}_{\theta_{k-1}}$ by $\mathbf{Z}_{\theta_{k-1}}^T \mathbf{Z}_{\theta_{k-1}}$ in Figure 1. We call this the *constrained fundamental numerical scheme* (CFNS). A necessary condition for CFNS to converge is that, for each $k = 0, 1, \dots$, the smallest (non-negative) eigenvalue of $\mathbf{Z}_{\theta_k}^T \mathbf{Z}_{\theta_k}$ should be sufficiently well separated from the remaining eigenvalues. When this condition is satisfied, the algorithm seeded with an estimate close enough to the sought-after constrained minimiser will produce updates quickly converging to that minimiser. Interestingly, many other, often simpler, equivalent forms of (8) like

$$[\mathbf{Y}_\theta \theta]_{\theta=\hat{\theta}_{\text{AML}}} = \mathbf{0} \quad \text{with} \quad \mathbf{Y}_\theta = \|\theta\|^2 \mathbf{P}_\theta \mathbf{X}_\theta \mathbf{P}_\theta + \mathbf{I}_l - \mathbf{P}_\theta$$

lead to non-converging algorithms, with divergence occurring irrespective of the distance of the initial estimate from the desired limit.

4 Experimental evaluation

In this section, we present results of comparative tests carried out to evaluate the performance of CFNS. Several algorithms, including CFNS, were used to compute the fundamental matrix from synthetic and real image data. Single data were formed by pairs

of corresponding points, the role of the principal constraint was played by the epipolar constraint, and the ancillary constraint was the condition that the determinant of the fundamental matrix should vanish. The covariances of the data were assumed to be default identity matrices corresponding to isotropic homogeneous noise in image point measurement.

The basic estimation methods considered were:

- **NALS** = Normalised Algebraic Least Squares Method
- **FNS** = Fundamental Numerical Scheme
- **CFNS** = Constrained FNS
- **GS** = Gold Standard Method.

Here, NALS refers to the *normalised* ALS method of Hartley [7], which takes suitably transformed data as input to ALS and back-transforms the resulting estimate; GS refers to the (theoretically optimal) bundle-adjustment, maximum-likelihood method described by Hartley and Zisserman [8], seeded with the FNS estimate; FNS and CFNS are as described earlier. CFNS was applied in the Hartley-normalised data domain. The data normalisation combined with back-transforming of estimates has no theoretical influence on the constrained minimiser, but in practice significantly improves separation of the smaller eigenvalues of the matrices $\mathbf{Z}_\theta^T \mathbf{Z}_\theta$ involved. The CFNS algorithm fails to converge when used with raw data, a phenomenon explained by the lack of sufficient eigenvalue separation.

When comparing the outputs of algorithms along various dimensions, it is critical that the ancillary constraint be perfectly satisfied. A convenient way to enforce this constraint is to correct an estimate of the fundamental matrix in a post-process. Any estimate $\hat{\mathbf{F}}$ with $\|\hat{\mathbf{F}}\| = 1$ can be modified to a rank-2 matrix $\hat{\mathbf{F}}_c$ with $\|\hat{\mathbf{F}}_c\| = 1$ by minimising the distance $\|\hat{\mathbf{F}} - \hat{\mathbf{F}}_c\|_F$ subject to the condition $\det \hat{\mathbf{F}}_c = 0$, where $\|\cdot\|_F$ denotes the Frobenius norm. The minimiser can easily be found by performing a SVD of $\hat{\mathbf{F}}$, setting the smallest singular value to zero and recomposing. For the estimate generated by FNS, a more sophisticated, Kanatani-like (cf. [9, Chap. 5]) correction can be obtained by means of the iterative process

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k - [\partial_\theta \phi(\boldsymbol{\theta}_k) \mathbf{H}_{\boldsymbol{\theta}_k}^- [\partial_\theta \phi(\boldsymbol{\theta}_k)]^T]^{-1} \phi(\boldsymbol{\theta}_k) \mathbf{H}_{\boldsymbol{\theta}_k}^- [\partial_\theta \phi(\boldsymbol{\theta}_k)]^T, \quad (10)$$

where $\mathbf{H}_{\boldsymbol{\theta}_k}^-$ denotes the pseudo-inverse of $\mathbf{H}_{\boldsymbol{\theta}_k}$.

Our computed estimates were usually post-hoc rank-2 corrected. In case of the NALS method, SVD correction preceded the final back-transformation of estimates. In the following, we use the notation “+” to denote a post-process SVD correction, and “++” to denote an iterative correction (see (10)) followed by SVD correction. Thus, the composition of FNS and SVD correction is denoted by FNS+. Of the various methods listed here, only SVD correction is guaranteed to generate a perfectly rank-2 estimate, although CFNS, GS and the iterative correction usually get extremely close.

4.1 Synthetic image tests

Synthetic tests are valuable in comparative testing as we have ground truth available, and we may employ repeated trials yielding results of statistical significance.

The regime adopted was to generate true corresponding points for some stereo configuration and collect performance statistics over many trials in which random Gaussian

	J_{AML}	$ \phi $
FNS	50.18	1.56×10^{-13}
FNS+	57.48	0
CFNS	52.62	3.07×10^{-25}
CFNS+	52.62	0

Table 1: J_{AML} and $|\phi|$ values for FNS and CFNS before and after SVD rank-2 correction.

	J_{AML}	<i>Reproj. error</i>	<i>Time</i>
NALS+	57.50	1.278	0.02
FNS+	57.47	1.278	0.29
FNS++	53.42	1.265	0.61
CFNS+	52.62	1.263	0.23
GS+	52.62	1.263	3.50

Table 2: J_{AML} residuals, reprojection errors and execution times for rank-2 estimates.

perturbations were made to the image points. Many configurations were investigated and the results below are typical. Specifically, we conducted experiments by first choosing a realistic geometric configuration for the cameras. Next, 30 3D points were randomly selected in the field of view of both cameras, and were then projected onto 500×500 pixel images to provide “true” matches. For each of 200 iterations, homogeneous Gaussian noise with standard deviation of 1.5 pixels was added to each image point and the contaminated pairs were used as input to the various algorithms.

Table 1 examines the FNS and CFNS methods in terms of the cost function, J_{AML} , and the ancillary constraint residual, $|\phi|$. As is to be expected, and consistent with its design, FNS generates the smallest value of J_{AML} , but leaves a non-zero ancillary constraint value, $|\phi|$. CFNS reduces the value of $|\phi|$ almost to zero and (necessarily) incurs a small increase in J_{AML} . Note that a subsequent SVD-correction (which ensures $\phi = 0$) of the FNS estimate results in an associated J_{AML} value that is substantially increased. In contrast, SVD-correction of the CFNS estimate leaves the J_{AML} value virtually unaffected, and much smaller than the corrected FNS estimate. This test, which is typical, confirms that CFNS is operating as designed.

Table 2 compares the J_{AML} values generated by the methods NALS+, FNS+, FNS++, CFNS+, and GS+. Note that all of the methods undergo a final SVD-rank-2-correction ensuring that the ancillary constraint is perfectly satisfied. Were we to avoid this step (in, say, the CFNS and GS approaches) it might be unclear whether a low J_{AML} value was due to the constraint not having been fully satisfied.

The results show that, with respect to J_{AML} , GS+ and CFNS+ perform best and equally well, with FNS++ only a little behind; FNS+ and NALS+ are set further back. The same ordering occurs when using a measure in which the estimated fundamental matrix is employed to reproject the data and compute the distance of the data from the truth. This reprojection-error from truth may be regarded as an optimal measure in the synthetic

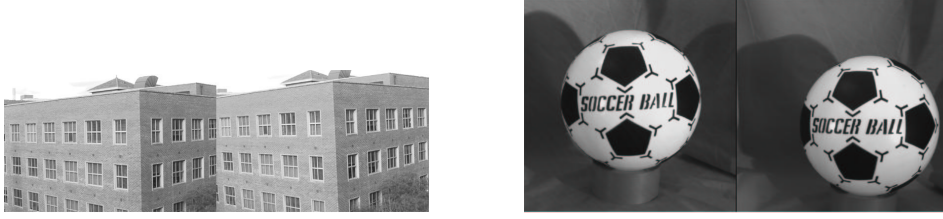


Figure 2: The building and soccer ball stereo image pairs.

realm.

Finally, a timing test is also presented in Table 2. Here we give the average time over 100 trials to compute NALS, FNS, CFNS, and GS. Unsurprisingly, GS turns out to be by far the slowest of the methods. While it may be speeded up via the incorporation of sparse-matrix techniques, it is destined to be relatively slow given the high-dimensionality of the search.

CFNS thus emerges as an excellent means of estimating the fundamental matrix. Its performance is commensurate with GS while being much faster. FNS++ is only a little short of CFNS in speed and accuracy. However, it does not have the advantage of being an *integrated* method of constrained minimisation.

4.2 Real image tests

The image pairs from which we estimate fundamental matrices are presented in Figures 2. They exhibit variation in subject matter, and in the camera set-up used for acquisition. Features were detected in each image using the Harris corner detector [6]. A set of corresponding points was generated for each image pair by manually matching the detected features. The number of matched points was 44 for the building, and 55 for the soccer ball. For each estimation method, the entire set of matched points was used to compute a fundamental matrix.

Each estimator was used to generate a fundamental matrix. Tables 3 and 4 show results obtained for various methods when dealing with the soccer-ball and building images, respectively. Measures used for comparison are J_{AML} and the reprojection error to data (the distance between the reprojected data and the original data). Note that the ancillary constraint is in all cases perfectly satisfied. CFNS+ and GS+ give the best results and are essentially inseparable, while FNS++ is only slightly behind. FNS+ and NALS+ lag much further behind.

5 Conclusion

We presented a short experimental study to evaluate the performance of a newly designed constrained estimator, CFNS. Our study indicates that CFNS produces estimates satisfying the imposed constraint, with values of the underlying cost function no greater than those generated by other methods. CFNS generates results of similar accuracy to those generated by the Gold Standard Method, but in a fraction of the time. Furthermore, CFNS

	J_{AML}	Reproj. error (to data)
NALS+	0.799	0.0926
FNS+	0.813	0.0933
FNS++	0.422	0.0681
CFNS+	0.442	0.0681
GS+	0.442	0.0681

Table 3: J_{AML} residuals and reprojection errors for rank-2 estimates - soccer ball images.

	J_{AML}	Reproj. error (to data)
NALS+	5.35	0.285
FNS+	4.88	0.275
FNS++	2.05	0.173
CFNS+	1.88	0.163
GS+	1.88	0.163

Table 4: J_{AML} residuals and reprojection errors for rank-2 estimates - building images.

has the advantages over FNS++ of being a genuinely integrated scheme for constrained minimisation, and producing slightly more accurate results.

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